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Optimal Pairs Trading Strategies: A Stochastic Mean–Variance Approach

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Abstract
In this paper, we consider optimal pairs trading strategies in terms of static optimality and dynamic optimality under mean–variance criterion. The spread of the entity pairs is assumed to be mean-reverting and follows an Ornstein–Uhlenbeck process. A constrained optimal control problem is considered, and the Lagrange multiplier technique is adopted to transform the primal problem into a family of linear-quadratic optimal control problems that can be solved by the classical dynamic programming principle. Both solutions for static and dynamic optimal pairs trading problems are derived and discussed. We show that the "static and dynamic optimality" is a viable approach to the time-inconsistent control problem. Furthermore, numerical experiments are presented to demonstrate the performance of the optimal pairs trading strategies.

Keywords Pairs trading · Mean–variance (MV) analysis · Time inconsistency · Dynamic optimality · Ornstein–Uhlenbeck (OU) process

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1 Introduction

Pairs trading is a trading strategy on highly correlated stock pairs whose prices move together historically and predictably. After being applied to industry by Nunzio Tartaglia's team at Wall Street in mid 1980s, pairs trading has become popular as a market-neutral investment strategy and has been widely used by traders and hedge funds. There are three main techniques that can be utilized to execute a pairs trading strategy: distance method, cointegration method and stochastic spread method. For more discussion on classification of pairs trading methods, please refer to [14] and [17]. The distance method attempts to make profits when the squared price difference between the concerned asset pairs triggers a prescribed level, see [23]. This method is straightforward; however, it lacks the ability to forecast the convergence time and holding period that determine the success of a pairs trading strategy. To address this issue, [31] pioneered the cointegration approach which assumes that the price dynamics of the asset pair are cointegrated. Cointegration is a statistical relationship where two integrated time series can be linearly combined to produce a single stationary time series. By applying this method, the spread of the integrated pairs, which could be modeled as the logarithm difference of a pair of stock prices, becomes tractable. The literature that is categorized as using the cointegration method includes [11, 19, 22, 31] and [30], etc. However, the stochastic spread method explicitly models the price spread between the paired assets. The spread is commonly modeled as a mean-reverting process, for instance, the Cox–Ingersoll–Ross process or the OU process. Papers based on this approach include [6, 9, 27] and [4], etc. In addition to these three main methods, [15, 16] considered another method which is a sort of combined forecast using Multi-Criteria Decision Methods. However, no other research work has followed this method so far. In this paper, we adopt the stochastic spread method.

We investigate the optimal pairs trading problem under the MV criterion pioneered by [21]. It has long been commonly used in both academia and industry because of its simplicity and intuitive appeal. The objective of investors can be intuitively expressed in the MV criterion as maximizing the expected terminal wealth and controlling the investment risk with a risk aversion level represented by a parameter in front of variance in the problem. However, there is a time-inconsistency issue when we consider the pairs trading problem with MV objective using the stochastic spread model. The nonlinearity caused by variance in MV problem makes the DP approach no longer applicable. Well-known methods to deal with this time-inconsistent optimal control problem can be divided into two main categories. The first one is the pre-committed strategy. This method interprets the optimal as the optimal from the initial time, where the optimality is called the static optimality. [26] first adopted this method to MV analysis in a continuous-time setting. Afterward, discrete-time cases and those involving transaction cost are solved consequently, see [2, 18, 33, 34], and [5]. The second method is called the game theoretic approach, which addresses the time-inconsistent problem with subgame perfect Nash equilibrium. The idea of this method was first introduced by Stroz (1955) when studying the deterministic Ramsey problem, and the corresponding optimality concept was proposed by Selten in 1965. Further studies on continuous-time and discrete-time cases are done by [25] and [12], etc. As more researchers become interested in this topic, [7] and [8] firstly provided a precise definition of game theoretic equilibrium under the continuous time case. When it comes to the particular case of MV analysis, [1] used the total variance formula under game theoretic approach to solve this time-inconsistent problem.

In this paper, we consider optimal pairs trading strategies in terms of static and dynamic optimality under the MV criteria. The definitions of static and dynamic MV optimality are defined in [24]. They proposed a new methodology for solving nonlinear optimal control problems and demonstrated it in MV analysis. This new approach considered the related constrained problems and made use of Lagrange multipliers to transform nonlinear problems into a family of LQ problems, which can be solved by using the classical Hamilton–Jacobi–Bellman (HJB) approach. We adopt this new method in MV analysis and apply it to pairs trading. To obtain the dynamic optimality, we change the initial time and initial value of wealth in the static optimal solution to any time afterward and the corresponding value of wealth at that time, respectively. We then show that this new solution is indeed the dynamic optimal solution.

[35] studied the optimal pairs trading strategies with MV criterion. They compared the two cases of "symmetric" and "non-symmetric" trading constraints. By employing the equilibrium strategy in [1], they derived the analytical solutions to the optimal control problems under two constraint cases, respectively. In this work, we adopt the same price spread process as [35], but work on solutions with static and dynamic MV optimalities, in the same spirit as [24]. We obtained the static optimal strategies and then extended them to the dynamic optimality successfully. The key conceptual difference between these two papers is that the definition of equilibrium optimality is constrained in the sense that the optimal control at time t is best among all "available" control, while the definition of dynamic optimality is unconstrained in the sense that the optimal control at time t is best among all "possible" controls afterward. For more details, we refer readers to Section 4 of [24].

The paper is structured as follows. Section 2 introduces the model dynamics for a pairs trading problem in a continuous-time setting and the concerned MV optimal control problem. Section 3 then discusses how to solve the control problem by introducing the constrained problem and using Lagrange multiplier techniques. The detailed derivation of the analytical static optimal solution is provided. Section 4 further derives the dynamic optimal solution to the pairs trading control problem. Details of the validation and explanations are also provided. Section 5 presents simulated numerical experiments to illustrate the application of the optimal pairs strategy obtained. Finally, Sect. 6 concludes the paper.

2 The Model Setup

Assume that a risk-free asset $N(t)$ exists with a risk-free rate of r compounded continuously. Thus, $N(t)$ satisfies the dynamics

$$\mathrm{d}N(t) = rN(t)\mathrm{d}t. \tag{1}$$

We adopt [22]'s framework to setup the pair trading model. Let $A(t)$ and $B(t)$ denote, respectively, the prices of stocks A and B at time t. We assume that stock B follows a geometric Brownian motion

$$\frac{\mathrm{d}B(t)}{B(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}Z(t), \tag{2}$$

where constants μ and σ are the drift and the volatility, respectively. Here, $Z(t)$ is a standard Brownian motion. Let γ be a predetermined ratio[1] and $X(t)$ denote the spread of stocks A and B at time t, defined as

$$X(t) = \ln(A(t)) - \gamma \ln(B(t)). \tag{3}$$

We further assume that the spread follows an OU process

$$\mathrm{d}X(t) = k(\theta - X(t))\mathrm{d}t + \eta \mathrm{d}W(t), \quad X(t_0) = x_0, \tag{4}$$

where $W(t)$ is a standard Brownian motion, and ρ denotes the instantaneous correlation coefficient between $Z(t)$ and $W(t)$. Therefore, by a straightforward calculation, we obtain

$$\frac{\mathrm{d}A(t)}{A(t)} = \left[k(\theta - X(t)) + \gamma\mu + \frac{\eta^2}{2} + \frac{\sigma^2\gamma(\gamma-1)}{2} + \rho\sigma\eta\gamma\right]\mathrm{d}t + \sigma\mathrm{d}Z(t) + \eta\mathrm{d}W(t). \tag{5}$$

Let $V(t)$ be the value of a self-financing pairs-trading portfolio, $h(t)$ and $1-(1-\gamma)h(t)$ denote the portfolio weights of the stock pair and the risk-free asset at time t, respectively. Then, the wealth process $V(t)$ becomes

$$\mathrm{d}V(t) = V(t)\left[h(t)\frac{\mathrm{d}A(t)}{A(t)} - h(t)\gamma\frac{\mathrm{d}B(t)}{B(t)} + [1-(1-\gamma)h(t)]\frac{\mathrm{d}N(t)}{N(t)}\right], \tag{6}$$

and substituting Eqs. (2) and (5) into Eq. (6), we have

$$\mathrm{d}V(t) = V(t)\left\{[r + (\kappa - kX(t))\,h(t)]\mathrm{d}t + \eta h(t)\mathrm{d}W(t)\right\}, \quad V(t_0) = v_0, \tag{7}$$

where

$$\kappa = k\theta + \frac{\sigma^2\gamma(\gamma-1)}{2} + \frac{1}{2}\eta^2 + \rho\sigma\eta\gamma - (1-\gamma)r. \tag{8}$$

Remark 1 If we further consider more general dynamics of stock B, for instance, we consider geometric Lévy process to include jumps, then the dynamics of B can be written by

$$\frac{\mathrm{d}B(t)}{B(t-)} = \mu \mathrm{d}t + \sigma dZ(t) + d\left(\sum_{i=1}^{K(t)}\left(e^{\xi_i} - 1\right)\right), \tag{9}$$

where $K(t)$ is a Poisson process with intensity ν, and ξ_i for $i = 1, 2, \ldots$ are random variables that model the sizes of jumps. Note that we assume the sizes to be $e^{\xi_i} - 1 > -1$ in order to ensure that $B(t) > 0$.

[1] In practice, the value of γ is important to the pairs trading strategies and can be determined by co-integration test. For more detailed discussion, we refer to [10]. If the two securities are stocks from the same financial sector (like two banking stocks), one may take this ratio to be unity.

With the definition given by Eq. (3) and the assumption given by Eq. (4), we then can derive the dynamics of stock A accordingly. Assume the j-th jump occurred in the stock price B at time t, then the jump size at t is $\Delta B(t) = B(t) - B(t-) = (e^{\xi_j} - 1)B(t-)$. We therefore have

$$\begin{aligned}\Delta(\ln(B))(t) &= \ln(B(t-) + \Delta B(t)) - \ln(B(t-)) \\ &= \ln(B(t-)(1 + e^{\xi_j} - 1)) - \ln(B(t-)) \\ &= \xi_j.\end{aligned} \tag{10}$$

Since the spread X follows an OU process, there is no jump in X. With $\ln(A(t)) = X(t) + \gamma \ln(B(t))$, we know that a jump in A only occurs when there is a jump in B. Hence,

$$\Delta(\ln(A))(t) = \gamma \Delta(\ln(B))(t) = \gamma \xi_j. \tag{11}$$

We then can derive the jump size in stock A by

$$\begin{aligned}\Delta A(t) = A(t) - A(t-) &= e^{\ln(A(t))} - e^{\ln(A(t-))} \\ &= e^{\ln(A(t-)+\Delta(\ln(A))(t))} - e^{\ln(A(t-))} \\ &= A(t-)(e^{\Delta(\ln(A))(t))} - 1) \\ &= A(t-)(e^{\gamma\xi_j} - 1).\end{aligned} \tag{12}$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}A(t)}{A(t-)} = &\left[k(\theta - X(t)) + \gamma\mu + \frac{\eta^2}{2} + \frac{\sigma^2\gamma(\gamma-1)}{2} + \rho\sigma\eta\gamma\right]\mathrm{d}t + \gamma\sigma dZ(t) + \eta \mathrm{d}W(t) \\ &+ d\left(\sum_{i=1}^{K(t)}\left(e^{\gamma\xi_i} - 1\right)\right).\end{aligned} \tag{13}$$

The corresponding dynamics of the wealth process V then become

$$\begin{aligned}\mathrm{d}V(t) = V(t-)&\left\{[r + (\kappa - kX(t))\, h(t)]\mathrm{d}t + \eta h(t)\mathrm{d}W(t)\right\}, \\ &+ h(t)d\left(\sum_{i=1}^{K(t)}\left(e^{\gamma\xi_i} - 1\right)\right) - h(t)\gamma d\left(\sum_{i=1}^{K(t)}\left(e^{\xi_i} - 1\right)\right).\end{aligned} \tag{14}$$

Note that the jump terms in the dynamics of V cancel when $\gamma = 1$, but not for $\gamma \neq 1$. In short,

- If $\gamma = 1$, the dynamics of V stay the same as given in Eq. (7) even when jumps generated by a geometric Lévy process are included in the dynamics of B. Hence, the trading strategies derived in this paper are still applicable in this case.
- If $\gamma \neq 1$, there will be jump terms in the dynamics of V as given in Eq. (14). In this case, the optimal control problem becomes significantly more complicated and is beyond the scope of this paper.

We consider the optimal control problem

$$J(t, x, v) = \sup_h \{E_{t,x,v}[V^h(T)] - cVar_{t,x,v}[V^h(T)]\}, \tag{15}$$

where the supremum is taken over all admissible controls h such that $E_{t,x,v}[V^h(T)^2] < \infty$ for $(t, x, v) \in [t_0, T] \times \mathbb{R} \times \mathbb{R}$ and $c > 0$ is a given and fixed constant which reflects the investor's risk aversion level. Here, x and v represent the values of $X(t)$ and $V^h(t)$ at time $t \in [t_0, T]$, respectively.

3 Solution to the Problem

We first derive the static optimality for the problem. Let $\mathbb{E}_0$ denote the expectation E_{t_0,x_0,v_0}. Notice that

$$\mathbb{E}_0[V^h(T)] - cVar_{t_0,x_0,v_0}[V^h(T)] = \mathbb{E}_0[V^h(T)] + c\mathbb{E}_0[V^h(T)]^2 - c\mathbb{E}_0[(V^h(T))^2]. \tag{16}$$

To overcome the difficulty of quadratic nonlinearity, we consider to fix

$$\mathbb{E}_0[V^h(T)] = M,$$

where $M \in \mathbb{R}$ is given, then

$$\begin{aligned} J(t_0, x_0, v_0) &= \sup_{M\in\mathbb{R}} \sup_{h:\mathbb{E}_0[V^h(T)]=M} \{\mathbb{E}_0(V^h(T)) - cVar_{t_0,x_0,v_0}[V^h(T)]\} \\ &= \sup_{M\in\mathbb{R}} \sup_{h:\mathbb{E}_0[V^h(T)]=M} \{\mathbb{E}_0(V^h(T)) + c\mathbb{E}_0[V^h(T)]^2 - c\mathbb{E}_0[(V^h(T))^2]\} \\ &= \sup_{M\in\mathbb{R}} \{M + cM^2 - c \inf_{h:\mathbb{E}_0[V^h(T)]=M} \mathbb{E}_0[(V^h(T))^2]\}. \end{aligned} \tag{17}$$

To begin with, we consider the constrained problem

$$J_M(t_0, x_0, v_0) = \inf_{h:\mathbb{E}_0[V^h(T)]=M} \mathbb{E}_0[(V^h(T))^2], \tag{18}$$

and we apply the Lagrange multipliers method to solve this problem. Define the Lagrangian as follows

$$L_{t_0,x_0,v_0}(h, \lambda) = \mathbb{E}_0[(V^h(T))^2] - \lambda \left(\mathbb{E}_0[V^h(T)] - M\right) \tag{19}$$

for $\lambda \in \mathbb{R}$. Denote h^*_λ to be the optimal control in the unconstrained problem

$$L_{t_0,x_0,v_0}(h^*_\lambda, \lambda) := \inf_h L_{t_0,x_0,v_0}(h, \lambda). \tag{20}$$

We further suppose that

$$\mathbb{E}_0[V^{h^*_\lambda}(T)] = M, \tag{21}$$

for a specific $\lambda \in \mathbb{R}$, then we have

$$\mathbb{E}_0[(V^{h^*_\lambda}(T))^2] = L_{t_0,x_0,v_0}(h^*_\lambda, \lambda) \leq \mathbb{E}_0[(V^h(T))^2] \tag{22}$$

for any control h such that $\mathbb{E}_0[(V^h(T))] = M$. This means h^*_λ is the optimal control of the constrained problem in Eq. (18) when satisfying both Eqs. (20) and (21). Thus, to solve Problem (18), it suffices to solve Eq. (20) and Eq. (21). Therefore, we consider the following optimal control problem:

$$J^\lambda(t_0, x_0, v_0) := \inf_h \mathbb{E}_0[(V^h(T))^2 - \lambda V^h(T)]. \tag{23}$$

Using the classic DP approach, we have the HJB equation:

$$0 = \inf_h \left[J^\lambda_t + k(\theta - x)J^\lambda_x + \frac{\eta^2 J^\lambda_{xx}}{2} + [r + h(\kappa - kx)]vJ^\lambda_v + \frac{\eta^2 h^2 v^2 J^\lambda_{vv}}{2} + \eta^2 hvJ^\lambda_{xv} \right]. \tag{24}$$

Making the ansatz that $J^\lambda_{vv} > 0$ and minimizing the quadratic function, we have

$$(\kappa - kx)vJ^\lambda_v + \eta^2 v^2 hJ^\lambda_{vv} + \eta^2 vJ^\lambda_{xv} = 0, \tag{25}$$

which gives us

$$h^* = -\frac{\eta^2 J^\lambda_{xv} + (\kappa - kx)J^\lambda_v}{\eta^2 vJ^\lambda_{vv}}. \tag{26}$$

For simplicity of notation, we rewrite

$$h^* = -\frac{\eta J^\lambda_{xv} + \delta J^\lambda_v}{\eta vJ^\lambda_{vv}}, \tag{27}$$

where $\delta = \frac{\kappa - kx}{\eta}$. Substituting h^* back into Eq. (24), it reduces to

$$J^\lambda_t + k(\theta - x)J^\lambda_x + \frac{1}{2}\eta^2 J^\lambda_{xx} + rvJ^\lambda_v - \frac{(\delta J^\lambda_v + \eta J^\lambda_{xv})^2}{2J^\lambda_{vv}} = 0, \tag{28}$$

with $J^\lambda(T, X(T), V(T)) = V(T)^2 - \lambda V(T)$. In order to solve the above equations, we noted that $J^\lambda(T, X(T), V(T))$ has a quadratic growth in $V(T)$ by Eq. (28). Therefore, we assume the ansatz takes the following form:

$$J^\lambda(t, x, v) = a(t, x)v^2 + b(t, x)v + c(t, x). \tag{29}$$

Then, Eq. (28) becomes

$$\begin{aligned} &a_t v^2 + b_t v + c_t + k(\theta - x)(a_x v^2 + b_x v + c_x) + \frac{1}{2}\eta^2(a_{xx}v^2 + b_{xx}v + c_{xx}) + rv(2av + b) \\ &- \frac{1}{4a}[\delta^2(4a^2v^2 + 4abv + b^2) + 2\delta\eta(4aa_x v^2 + 2ab_x v + 2a_x bv + bb_x) + \eta^2(4a_x^2 v^2 + 4a_x b_x v + b_x^2)] = 0. \end{aligned} \tag{30}$$

We then have the system

$$
\begin{aligned}
&a_t + k(\theta - x)a_x + \tfrac{1}{2}\eta^2 a_{xx} + 2ra - \delta^2 a - 2\delta\eta a_x - \tfrac{\eta^2 a_x^2}{a} = 0,\\
&b_t + k(\theta - x)b_x + \tfrac{1}{2}\eta^2 b_{xx} + rb - \delta^2 b - \delta\eta b_x - \tfrac{\delta\eta a_x b}{a} - \tfrac{\eta^2 a_x b_x}{a} = 0,\\
&c_t + k(\theta - x)c_x + \tfrac{1}{2}\eta^2 c_{xx} - \tfrac{(\delta b + \eta b_x)^2}{4a} = 0,
\end{aligned} \tag{31}
$$

with

$$
\begin{aligned}
a(T, X(T)) &= 1,\\
b(T, X(T)) &= -\lambda,\\
c(T, X(T)) &= 0.
\end{aligned} \tag{32}
$$

By the assumption $J^{\lambda}_{vv} > 0$, we assume $a = e^P$. Then, from the first equation in Eq. (31), we have

$$
P_t + [k(\theta - x) - 2\delta\eta]P_x - \frac{1}{2}\eta^2 P_x^2 + \frac{1}{2}\eta^2 P_{xx} + 2r - \delta^2 = 0. \tag{33}
$$

Substituting δ into this equation and assuming $P(t, x) = m_1(t)x^2 + n_1(t)x + l_1(t)$, we obtain

$$
\begin{aligned}
&m_1' + 2km_1 - 2\eta^2 m_1^2 - \tfrac{k^2}{\eta^2} = 0,\\
&n_1' + (k - 2\eta^2 m_1)n_1 + 2(k\theta - 2\kappa)m_1 + \tfrac{2\kappa k}{\eta^2} = 0,\\
&l_1' + (k\theta - 2\kappa)n_1 - \tfrac{\eta^2 n_1^2}{2} + \eta^2 m_1 + 2r - \tfrac{\kappa^2}{\eta^2} = 0,
\end{aligned} \tag{34}
$$

with

$$
\begin{aligned}
m_1(T) &= 0,\\
n_1(T) &= 0,\\
l_1(T) &= 0.
\end{aligned} \tag{35}
$$

Notice that the first equation in Eq. (34) is a Riccati equation, and the solution of this equation is given as follows:

$$
m_1(t) = \frac{k}{2\eta^2}\left[1 + \xi - \frac{\xi^2 + 1}{\xi + \tan(kt)}\right], \tag{36}
$$

where

$$
\xi = \frac{1 - \tan(kT)}{1 + \tan(kT)}.
$$

With the solution of $m_1(t)$, both $n_1(t)$ and $l_1(t)$ can be solved accordingly and are given as follows:

$$
n_1(t) = e^{-I(t)}\left[\int \left(2(k\theta - 2\kappa)m_1(t) + \frac{2\kappa k}{\eta^2}\right) e^{I(t)}\mathrm{d}t + C_{n1}\right], \tag{37}
$$

$$
l_1(t) = (2\kappa - k\theta)tn_1(t) + \frac{\eta^2 t n_1^2(t)}{2} - \eta^2 t m_1(t) - 2rt + \frac{\kappa^2 t}{\eta^2} + C_{l1}, \tag{38}
$$

where constant C_{n1} and C_{l1} satisfy the terminal conditions and

$$I(t) = \int \left[k - 2\eta^2 m_1(t)\right] \mathrm{d}t = -\frac{k\xi t}{2\eta^2} + \ln|\tan(kt) + \xi| - \frac{\ln(\sec^2(kt))}{2} + \left(kt - \pi \left\lfloor \frac{kt + \frac{\pi}{2}}{\pi} \right\rfloor\right). \tag{39}$$

Notice that they are not functions of λ. If we assume $b > 0$ and $b = e^Q$ or assume $b < 0$ and $b = -e^Q$, then from the second equation in Eq. (31), we have

$$Q_t + [k(\theta - x) - \delta\eta]Q_x + \frac{1}{2}\eta^2(Q_x^2 + Q_{xx}) - \delta\eta P_x - \eta^2 P_x Q_x + r - \delta^2 = 0. \tag{40}$$

Substituting δ into this equation and assuming

$$Q(t, x) = m_2(t)x^2 + n_2(t)x + l_2(t),$$

we then obtain

$$\begin{aligned} &m_2' - 4\eta^2 m_1 m_2 + 2\eta^2 m_2^2 + 2km_1 - \tfrac{k^2}{\eta^2} = 0, \\ &n_2' + 2\eta^2(m_2 - m_1)n_2 + (k - 2\eta^2 m_2)n_1 + 2(k\theta - \kappa)m_2 - 2\kappa m_1 + \tfrac{2\kappa k}{\eta^2} = 0, \\ &l_2' + (k\theta - \kappa - \eta^2 n_1)n_2 + \tfrac{n_2^2\eta^2}{2} + m_2\eta^2 - \kappa n_1 + r - \tfrac{\kappa^2}{\eta^2} = 0, \end{aligned} \tag{41}$$

with

$$\begin{array}{llll} m_2(T) = 0 & & m_2(T) = 0 & \\ n_2(T) = 0 & \text{if } b > 0, & \text{or } n_2(T) = 0 & \text{if } b < 0. \\ l_2(T) = \ln(-\lambda) & & l_2(T) = \ln(\lambda) & \end{array}$$

Notice that the first two equations in Eq. (41) would degenerate to the first two equations in Eq. (34) if we set $m_2(t) = m_1(t)$ and $n_2(t) = n_1(t)$. Therefore, $m_1(t)$ and $n_1(t)$ given by Eqs. (36) and (37) are the solutions to the first two equations of Eq. (41) as well. In addition, observe that the left-hand side of the equation for l_2 in Eq. (41) is independent of λ and thus, we can separate λ and t in $l_2(t; \lambda)$. Let $\tilde{l}_2(t)$ be the solution of the third equation in Eq. (41). We obtain that

$$\tilde{l}_2(t) = (2\kappa - k\theta)tn_1(t) + \frac{\eta^2 t n_1^2(t)}{2} - \eta^2 t m_1(t) - rt + \frac{\kappa^2 t}{\eta^2} + C_{l2}, \tag{42}$$

with C_{l2} satisfying the terminal condition of $\tilde{l}_2(T) = 0$. Therefore, $\tilde{l}_2(t)$ is independent of λ and thus,

$$l_2(t; \lambda) = \begin{cases} \ln(-\lambda) + \tilde{l}_2(t), & \text{for } b > 0, \\ \ln(\lambda) + \tilde{l}_2(t), & \text{for } b < 0. \end{cases} \tag{43}$$

According to the ansatz of J^λ in Eq. (29), h^* can then be rewritten in the following form:

$$h^* = -\frac{\eta(2a_x v + b_x) + \delta(2av + b)}{2a\eta v}. \tag{44}$$

We can therefore conclude the results in the following proposition.

Proposition 1 *An optimal solution to the MV problem in Eq. (15) with the dynamic constraint in Eq. (7) is given by:*

$$h^*(t,x,v) = -\frac{[(2\eta^2 m_1(t) - k)x + n_1(t)\eta^2 + \kappa](2e^{l_1(t)}v - \lambda e^{\tilde{l}_2(t)})}{2\eta^2 v e^{l_1(t)}}, \tag{45}$$

where $m_1(t)$, $n_1(t)$, $l_1(t)$ and $\tilde{l}_2(t)$ are given in Eqs. (36), (37), (38) and (42).

Substituting the optimal control h^* given by Eq. (45) into Eq. (7), we then get

$$\begin{aligned} &\mathrm{d}V(t,X(t)) \\ &= [\alpha(t,X(t)) + \beta(t,X(t))V(t,X(t))]\mathrm{d}t + [\gamma(t,X(t)) + \nu(t,X(t))V(t,X(t))]\mathrm{d}W(t), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \alpha(t,X(t)) &= \tfrac{\lambda}{\eta}(\kappa - kX(t))g(t,X(t))e^{\tilde{l}_2(t)-l_1(t)}, \\ \beta(t,X(t)) &= r - \tfrac{2}{\eta}g(t,X(t))(\kappa - kX(t)), \\ \gamma(t,X(t)) &= \lambda g(t,X(t))e^{\tilde{l}_2(t)-l_1(t)}, \\ \nu(t,X(t)) &= -2g(t,X(t)), \end{aligned} \tag{47}$$

and

$$g(t,X(t)) = \frac{(2\eta^2 m_1(t) - k)X(t) + n_1(t)\eta^2 + \kappa}{2}. \tag{48}$$

Notice that the optimal strategy h^* and the corresponding optimal $V^{h^*}(T)$ depend on the value of λ which satisfies $E_0[V^{h^*_\lambda}(T)] = M$. According to Eq. (17), to find out $J(t,x,v)$, we need to find out the optimal M that maximize Eq. (17). For each value of M, there is a corresponding λ. However, there is no explicit expression for $E_{t,x,v}[V^h(T)] = M$. Therefore, we do not have the explicit expression of M in terms of λ. In order to explore all M to find out the optimal value that maximizes the control problem Eq. (15), it suffices to explore all λ to maximize the control problem Eq. (15). The corresponding M that maximize Eq. (15) would therefore be the optimal one. Note that it may happen that several different values of λ have the same corresponding value of M, but it does not matter since we only concern about the optimal value of M that maximizes the control problem Eq. (15). Therefore, we can explore the values of λ to maximize Eq. (15), and the corresponding optimal strategy h^* would then maximize the control problem.

4 Extension to Dynamic Optimal Control Problem

Replacing t_0, x_0 and v_0 with t, x and v, respectively, in the static optimal control $h^*_s := h^*$ as derived in the previous section, we can then obtain control h^*_d. We claim that this h^*_d is the optimal control for the dynamic case.

Proposition 2 *For every given and fixed $(t, x, v) \in [t_0, T] \times \mathbb{R} \times \mathbb{R}$, control h_d^* is defined to be $h_d^*(t, x, v) = h_s^*(t, x, v; \lambda(t, x, v))$. For any other admissible control $\bar{h}$ such that $\bar{h}(t, x, v) \neq h_d^*(t, x, v)$, we have*

$$J_{h_d^*}(t, x, v) := E_{t,x,v}[V^{h_d^*}(T)] - cVar_{t,x,v}[V^{h_d^*}(T)] \tag{49}$$

and

$$J_{\bar{h}}(t, x, v) := E_{t,x,v}[V^{\bar{h}}(T)] - cVar_{t,x,v}[V^{\bar{h}}(T)]. \tag{50}$$

Then, we claim that

$$J_{h_d^*}(t, x, v) > J_{\bar{h}}(t, x, v). \tag{51}$$

In other words, h_d^* is dynamically optimal in problem (15).

Proof By the definition of dynamically optimal control h_d^*,

$$J_{h_d^*}(t, x, v) = J_{h_s^*}(t, x, v; \lambda(t, x, v))$$

is the optimal value function if we view t, x, and v as the initial time conditions for statically optimal control h_s^*. Denote $M_{\bar{h}} := E_{t,x,v}[V^{\bar{h}}(T)]$, as we discussed before, we can rewrite $J_{\bar{h}}(t, x, v)$ as follows:

$$\begin{aligned} J_{\bar{h}}(t, x, v) &= M_{\bar{h}} + cM_{\bar{h}}^2 - cE_{t,x,v}[V^{\bar{h}}(T)^2] \\ &\leq M_{\bar{h}} + cM_{\bar{h}}^2 - cJ_{M_{\bar{h}}}(t, x, v). \end{aligned} \tag{52}$$

Similarly, we can rewrite

$$J_{h_d^*}(t, x, v) = M_* + cM_*^2 - cJ_{M_*}(t, x, v). \tag{53}$$

We now divide the proof into two cases: (i) $M_{\bar{h}} \neq M_*$ and (ii) $M_{\bar{h}} = M_*$.

(i) $M_{\bar{h}} \neq M_*$. Since M_* is the unique maximum point of the quadratic function in Eq. (17), the strict inequality follows.

$$\begin{aligned} J_{\bar{h}}(t, x, v) &\leq M_{\bar{h}} + cM_{\bar{h}}^2 - cJ_{M_{\bar{h}}}(t, x, v) \\ &< M_* + cM_*^2 - cJ_{M_*}(t, x, v) = J_{h_d^*}(t, x, v) \end{aligned} \tag{54}$$

Thus, Eq. (51) is proved under this case.

(ii) $M_{\bar{h}} = M_*$. To prove Eq. (51) under this case, we consider to prove the following strict inequality first:

$$J_{\bar{h}}^{\lambda^*}(t, x, v) := E_{t,x,v}[(V^{\bar{h}}(T))^2 - \lambda^* V^{\bar{h}}(T)] > J_{h_d^*}^{\lambda^*}(t, x, v), \tag{55}$$

where $J^{\lambda^*}_{h^*_d}(t,x,v)$ is defined in Eq. (23) and λ^* corresponds to the optimal control h^*. With Eqs. (24) and (28) and applying Ito's formula, we can obtain

$$\begin{aligned}
&(V^{\bar h}(T))^2 - \lambda^* V^{\bar h}(T)\\
&\quad = J^{\lambda^*}_{h^*_d}(T, X(T), V^{\bar h}(T))\\
&\quad = J^{\lambda^*}_{h^*_d}(t,x,v) + \int_t^T [J^{\lambda^*}_s + k(\theta - X(s))J^{\lambda^*}_x + [r + (\kappa - kX(s))\bar h]V^{\bar h}(s)J^{\lambda^*}_v\\
&\qquad + \tfrac{1}{2}\eta^2 J^{\lambda^*}_{xx} + \tfrac{1}{2}\eta^2\bar h^2 (V^{\bar h}(s))^2 J^{\lambda^*}_{vv} + \eta^2 \bar h V^{\bar h}(s) J^{\lambda^*}_{xv}]\mathrm{d}s + G_T,
\end{aligned} \tag{56}$$

where $G_s := \int_t^s [\eta J^{\lambda^*}_x + \eta \bar h V^{\bar h}(r) J^{\lambda^*}_v]\mathrm{d}W(r)$ is a martingale. Taking $E_{t,x,v}$ on both sides of Eq. (56), we obtain

$$\begin{aligned}
&J^{\lambda^*}_{\bar h}(t,x,v)\\
&\quad = J^{\lambda^*}_{h^*_d}(t,x,v) + E_{t,x,v}[\int_t^T [J^{\lambda^*}_s + k(\theta - X(s))J^{\lambda^*}_x + [r + (\kappa - kX(s))\bar h]V^{\bar h}(s)J^{\lambda^*}_v\\
&\qquad + \tfrac{1}{2}\eta^2 J^{\lambda^*}_{xx} + \tfrac{1}{2}\eta^2\bar h^2 (V^{\bar h}(s))^2 J^{\lambda^*}_{vv} + \eta^2 \bar h V^{\bar h}(s) J^{\lambda^*}_{xv}]\mathrm{d}s].
\end{aligned} \tag{57}$$

Notice that the integrand of Eq. (57) is non-negative according to Eq. (24). As we have already assumed that $\bar h(t,x,v) \neq h^*_d(t,x,v)$, and because of the continuity of $\bar h$ and h^*_d, we can find $\epsilon > 0$ which is small enough such that for all

$$(\tilde t, \tilde x, \tilde v) \in \mathbb{R}_\epsilon := [t, t+\epsilon] \times [x-\epsilon, x+\epsilon] \times [v-\epsilon, v+\epsilon],$$

$\bar h(\tilde t, \tilde x, \tilde v) \neq h^*_d(\tilde t, \tilde x, \tilde v)$ and $t+\epsilon < T$ hold. Since $h^*_d(t,x,v)$ is the minimum point of continuous function of Eq. (24) and $\bar h(\tilde t, \tilde x, \tilde v) \neq h^*_d(\tilde t, \tilde x, \tilde v)$ on $\mathbb{R}_\epsilon$. If ϵ is small enough, we can then find $p > 0$ such that

$$J^{\lambda^*}_s + k(\theta - \tilde x)J^{\lambda^*}_x + \frac{1}{2}\eta^2 J^{\lambda^*}_{xx} + [r + (\kappa - k\tilde x)\bar h]\tilde v J^{\lambda^*}_v + \frac{1}{2}\eta^2 \bar h^2 \tilde v^2 J^{\lambda^*}_{vv} + \eta^2 \bar h \tilde v J^{\lambda^*}_{xv} \geq p > 0. \tag{58}$$

Setting $\tau_\epsilon := \inf\{s \in [t, t+\epsilon] | (s, X(s), V^{\bar h}(s)) \notin \mathbb{R}_\epsilon\}$. Since both $X(t)$ and $V^{\bar h}(t)$ are continuous, then $\tau_\epsilon > t$. Therefore, from Eqs. (57) and (58) we obtain that

$$J^{\lambda^*}_{\bar h}(t,x,v) \geq J^{\lambda^*}_{h^*_d}(t,x,v) + pE_{t,x,v}[\tau_\epsilon - t_0] > J^{\lambda^*}_{h^*_d}(t,x,v), \tag{59}$$

which means the strict inequality in Eq. (55) holds. Recall Eqs. (18) to (23), we have

$$J^{\lambda^*}_{h^*_d}(t,x,v) = J_{M_*}(t,x,v) - \lambda^* M_*. \tag{60}$$

Under this case, we have $E_{t,x,v}[V^{\bar h}(T)] = M_{\bar h} = M_*$. Then, from Eq. (55), we further have

$$E_{t,x,v}[(V^{\bar h}(T))^2] - \lambda^* M_* = J^{\lambda^*}_{\bar h}(t,x,v) > J^{\lambda^*}_{h^*_d}(t,x,v) = J_{M_*}(t,x,v) - \lambda^* M_*. \tag{61}$$

Therefore, we have

$$E_{t,x,v}[(V^{\bar h}(T))^2] > J_{M_*}(t,x,v). \tag{62}$$

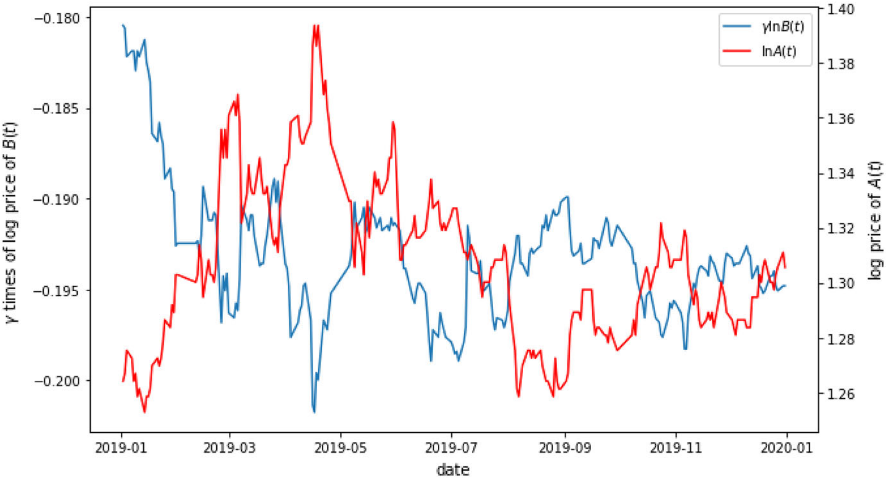


Fig. 1 The log price of the pair of stocks. Stock $A(t)$ is Bank of Communications (601328.SS), while stock $B(t)$ is Ping An Bank (000001.SZ). The pair ratio $\gamma = -0.0985$ is calculated from least square regression

Table 1 Estimated values for parameters in the model

Parameters	x_0	σ	μ	k	θ	η	ρ	γ
Values	2.0039	0.3081	0.6321	− 0.6263	1.9563	1.1240	0.1176	− 0.0985

Recall Eq. (17), we have

$$\begin{aligned} J_{h_d^*}(t,x,v) &= M_* + cM_*^2 - cJ_{M_*}(t,x,v) \\ &> M_* + cM_*^2 - cE_{t,x,v}[(V^{\bar{h}}(T))^2] \\ &= E_{t,x,v}[V^{\bar{h}}(T)] - cVar_{t,x,v}[V^{\bar{h}}(T)] \\ &= J_{\bar{h}}(t,x,v). \end{aligned} \tag{63}$$

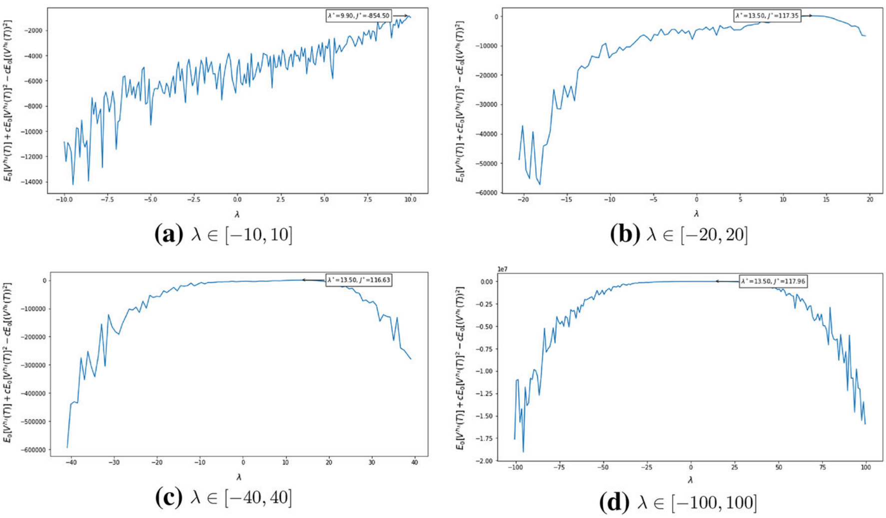
This shows Eq. (51) holds when $M_{\bar{h}} = M_*$.

We then conclude that Eq. (51) always holds and $h_d^*(t,x,v)$ is the dynamic optimal control. □

5 Numerical Experiments

In this section, we demonstrate how to apply the formulas and propositions in our paper to solve problems in practice. In the following numerical experiments, daily closing prices are employed. We choose a pair of highly co-integrated stocks,[2] namely, Ping An Bank (000001.SZ) and Bank of Communications (601328.SS) traded on the Chinese

[2] The formation is not the focus of this paper, so we omit the details here. For more discussions about the selection of pair of assets, please refer to [10, 11] and [13].



(a) $\lambda \in [-10, 10]$ **(b)** $\lambda \in [-20, 20]$

(c) $\lambda \in [-40, 40]$ **(d)** $\lambda \in [-100, 100]$

Fig. 2 Simulation results for the four chosen ranges

Table 2 Optimal value of λ under different ranges

Range of λ	[−10, 10]	[−20, 20]	[−40, 40]	[−100, 100]
Optimal λ	9.9	13.5	13.5	13.5

securities market from January 1, 2019 to December 31, 2019. The dynamics of log prices of these two stocks are presented in Fig. 1.

We set the initial wealth $v_0 = 100$, risk aversion factor $c = 0.2$ and interest rate $r = 5\%$. We consider the time length to be $T = 2$ years, which means the testing trading time horizon is from January 1, 2020 to December 31, 2021. Other parameters in the model are estimated using the real market data of the stock pairs: $000001.SZ$ and $601328.SS$, from January 1, 2019 to December 31, 2019. The calibrated parameters are given in Table 1. The calibration method is adopted from [22] where analytical formulas for the parameter estimates could be found in "Appendix" there. Note that there is no restriction on the selection of asset A and B, we adopt the best choice of asset A and B in the model according to the calibration.

In the following numerical experiments, we try to find the optimal λ in our model to maximize the control problem (15). To find the optimal value of λ, we consider four ranges of λ and calculate it's corresponding optimal value of control problem Eq. (15) by simulation. Based on the selection of the initial value, the four ranges of λ are chosen as [−10, 10], [−20, 20], [−40, 40] and [−100, 100]. The optimal value of λ and the corresponding J^* under four different ranges are presented in Fig. 2 and concluded in Table 2. Therefore, we obtain the optimal weight $h^*(t)$ by substituting $\lambda^* = 13.5$ into Eq. (45).

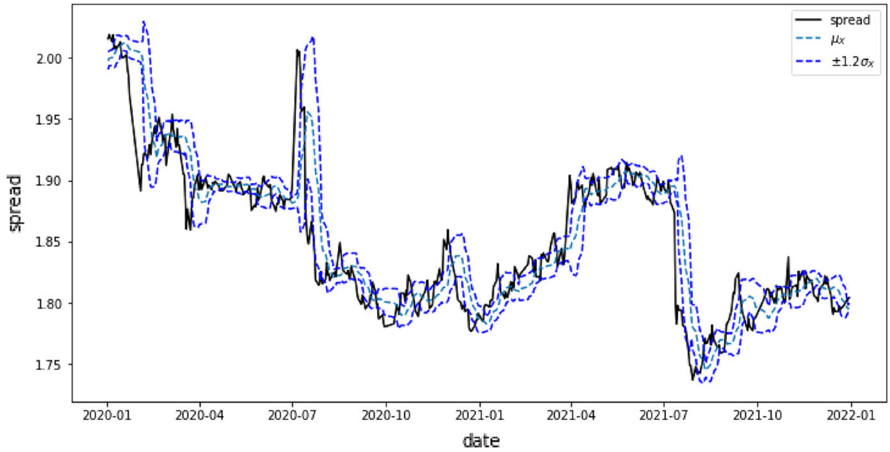


Fig. 3 Bollinger band with moving window size 10-day

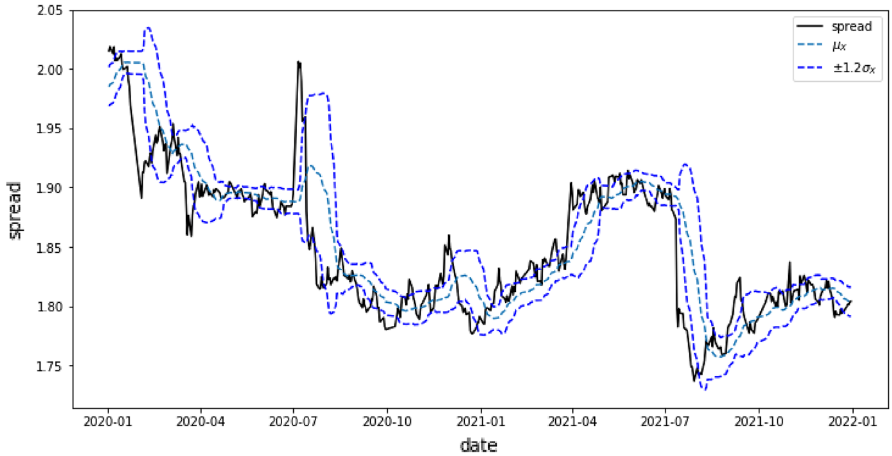


Fig. 4 Bollinger band with moving window size 20-day

To assess the performance of our MV pairs trading strategy (MVPT), we compare it with the normalized MVPT (nMVPT) and two other trading strategies which are chosen as benchmarks including

(i) The nMVPT:
Let $h_{\mathrm{MVPT}}(t) = h(t)$ denote the MVPT at time t. The nMVPT at time t is then defined as

$$h_{\mathrm{nMVPT}}(t) = \frac{h_{\mathrm{MVPT}}(t)}{\mathrm{abs}(\max_{s\in[0,T]} h_{\mathrm{MVPT}}(s))}, \quad t \in [0, T]. \tag{64}$$

(ii) Pairs trading strategy (PT) based on z-score:
This pairs trading idea is based on Bollinger bands in Bollinger (1992): when the

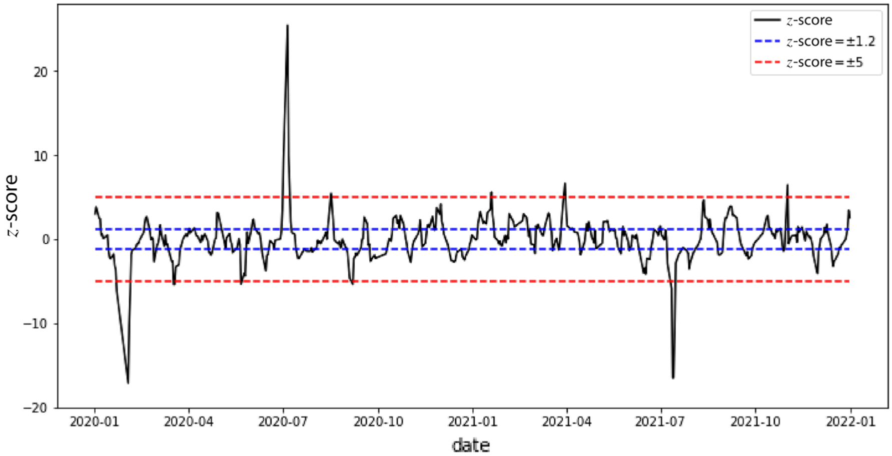


Fig. 5 z-score with moving window size 10-day

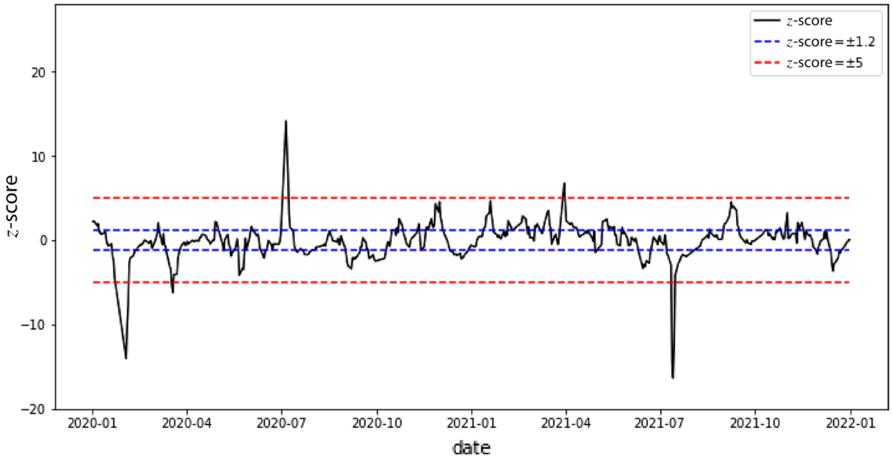


Fig. 6 z-score with moving window size 20-day

spread process hits the entry threshold, we open the positions on the pairs with $h(t)$ units, when it hits the exit threshold, we unwind the holding positions. We set the investment weight in the pair of stocks to be $h(t) = -\text{sign}(z\text{-score}(t))$, where

$$z\text{-score}(t) = \frac{X(t) - \mu_X(t)}{\sigma_X(t)}, \tag{65}$$

μ_X and σ_X are the time-dependent mean and standard deviation of spread process. We calculate μ_X and σ_X based on rolling historical data with window sizes 10-day and 20-day, respectively. In our experiments, we choose the entry, exit and

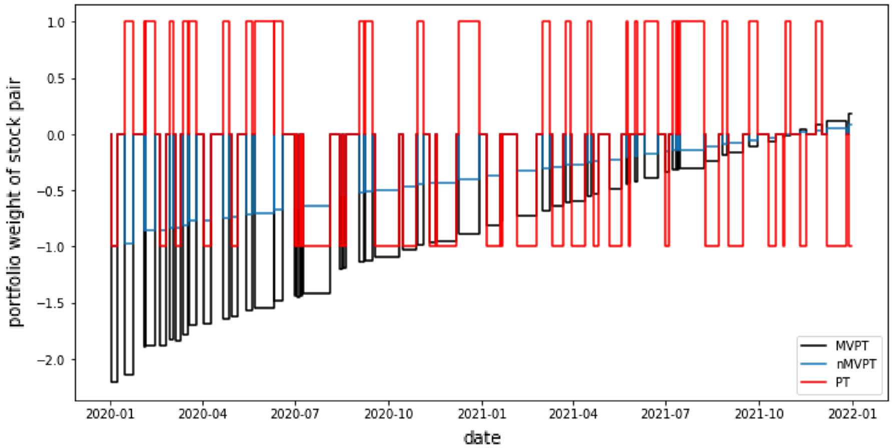


Fig. 7 Investment weight $h(t)$ with moving window size 10-day

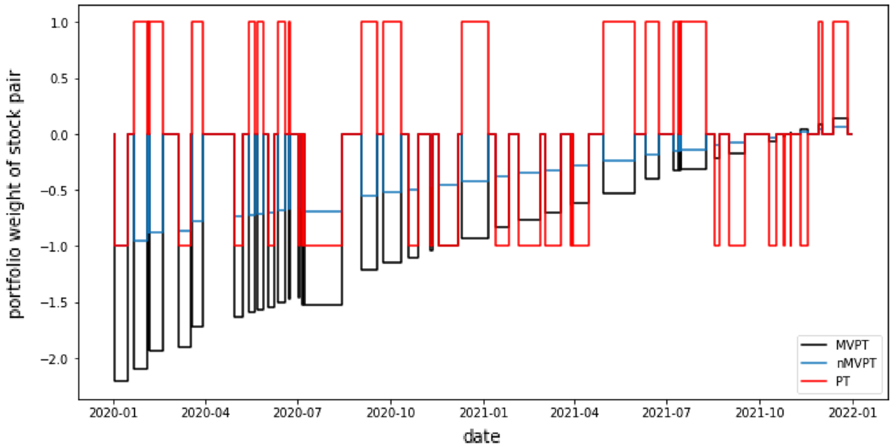


Fig. 8 Investment weight $h(t)$ with moving window size 20-day

stop-loss signals to be $\{|z\text{-score}| > 1.2\}$, $\{|z\text{-score}| < 0.5\}$ and $\{|z\text{-score}| > 5\}$, respectively.

(iii) Bond-only strategy (Bank):
In this case, we only consider investing in the risk-free asset. Therefore, the investment weight in the pair of stocks is set to be $h(t) = 0$.

We present the Bollinger band and z-score with different moving window sizes in Figs. 3, 4, 5 and 6. The different weights invested in the stock pair under MVPT, nMVPT and PT are shown in Figs. 7 and 8. The corresponding wealth processes under our trading strategy MVPT compared to nMVPT, PT and Bank are demonstrated in Figs. 9 and 10.

From Figs. 7 and 8, we notice that the weight invested in the pair of stocks under our MVPT strategy decreases as the investment time approaches to the terminal time

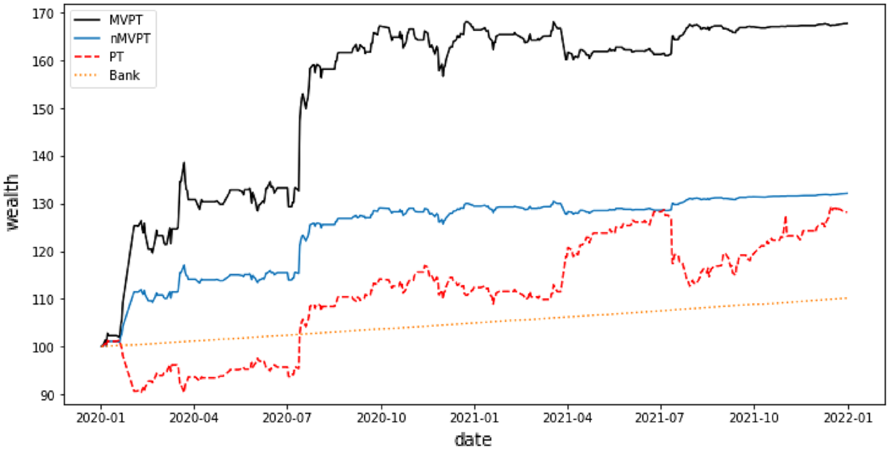


Fig. 9 Wealth processes with moving window size 10-day

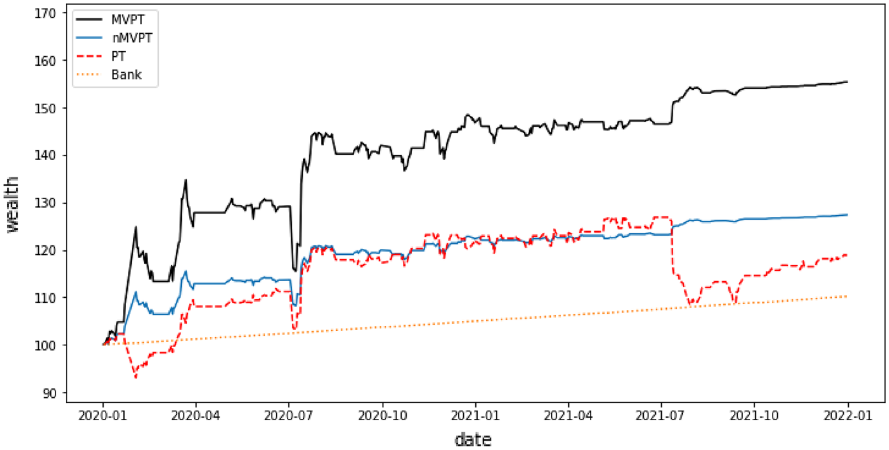


Fig. 10 Wealth processes with moving window size 20-day

T. This provides us the insight that investors tend to hold smaller positions in the pair of risky assets as time goes by to prevent risk. The wealth processes under four trading strategies in Figs. 9 and 10 show that our MVPT strategy outperforms the other three strategies with either 10-day or 20-day running mean and variance, which demonstrate the good performance of our trading strategy. In addition, the nMVPT outperforms the PT in terms of terminal wealth in both cases. Although with the 20-day moving window size, sometimes the PT slightly outperforms the nMVPT within the investment period, the performance of nMVPT is more stable over the whole investment period. The numerical results also illustrate the advantages of the MVPT strategy when the investment weights are not limited to remain between -1 and 1.

6 Conclusions

This paper considers optimal pairs trading strategies under the MV framework. The spread of the paired stocks is assumed to be mean-reverting and follows an OU process. To cope with the time-inconsistency, we work on solutions with static and dynamic MV optimalities, in the same spirit as [24]. This new approach introduced the related constrained problems and made use of Lagrange multipliers to deduce the nonlinear problems to a family of LQ problem, which can be solved by using classical HJB approach. We obtained the static optimal strategies and then successfully extended them to the dynamic optimality. In our numerical example, we illustrate the application of optimal pairs trading strategy, the way to find out the Lagrangian multiplier λ and demonstrate the good performance of our optimal trading strategy compared to other strategies.

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