IMA Journal of Management Mathematics (2020) **00**, 1–25
doi:10.1093/imaman/dpaa013

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How correlation risk in basket credit derivatives might be priced and managed?

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[Received on 18 September 2018; accepted on 5 May 2020]

In this paper, we construct quantitative models in which the dependence structure of the firms' default times is incorporated. Such models serve as the underlying frameworks in our proposed approach to price and hedge basket credit derivatives. Through the Gaussian copula-based method, we model the default correlation risk and develop valuation formulas for credit derivatives. Using single-name derivatives in a hedging strategy for basket credit derivatives, the utility of the delta and delta-gamma hedging techniques are examined. This enables the management of risk attributed to the changes in correlation without the need for a large number of hedging instruments. Our research contributions provide insights on how dependent risks in basket credit derivatives could be dealt with effectively.

Keywords: hedging, pricing, Gaussian copula models, basket credit default swaps.

1. Introduction

The credit crunch during the global financial crisis of 2008 has revealed the requirement for sound and practically useful methodologies for the pricing, hedging and risk management of various credit derivative products. Credit derivatives, whose payoffs are contingent on default events of reference entities, play a significant role in credit markets. As the credit derivatives market has been growing rapidly before the credit crunch of 2008 and quite steady after the crunch, various instruments have been innovated, e.g. credit default swaps (CDSs), collateralized debt obligations (CDOs) and credit linked notes (CLNs), etc. Among these credit derivatives, CDSs are considered as the fundamental product in the credit derivative market and they have been actively traded in the market.

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Credit risk models could be mainly classified into two major categories: structural firm-value model that is pioneered by Black & Scholes (1973) and Merton (1974) and reduced-form model by Jarrow & Turnbull (1995) and Madan & Unal (1998). In a structural firm-value model, a default is said to be triggered when the asset value falls below a certain threshold barrier. In a reduced-form model, the default process is supposed to be exogenous and the occurrences of defaults are modeled by random point processes such as Poisson processes. Both of the two types of credit risk models are quite popular as their own characteristics. More detailed discussion of these two types of models could be found in Cathcart & El-Jahel (2004). Although the structural firm-value models have intuitive link with the underlying value of the firm, in practice, the asset value of a firm is not observable continuously while the discretely observed data contains accounting noise. On the other hand, the reduced-form models would be more flexible and tractable. For this reason, we shall adopt a version of the reduced-form model in what follows. For pricing and hedging of credit derivatives (see, e.g. Dong *et al.*, 2018 and Ge *et al.*, 2015), an important issue is how to capture the dependent structure of defaults. The main approaches to model default dependence within various reduced-form models can be found in Li (2000), Davis & Lo (2001), Duffie & Gârleanu (2001), Jarrow & Yu (2001), Schönbucher (2003), Laurent & Gregory (2005), Rapisarda *et al.* (2007) and Yu (2007). Among these approaches, the factor Copula models are popular market standard for pricing credit derivatives (Frey & Backhaus, 2008). We shall discuss the pricing of basket CDSs conditional on survivorship information under different cases based on Gaussian copula model. We remark that the idea and method may be extended to general Copula models, e.g. Archimedean copulas.

The hedging of credit derivatives has been highlighted in recent years. The most commonly used hedging approach has been to 'delta hedge' the sensitivity shifts in risk factors using some, possibly simpler and more liquidly traded credit risky products for credit derivatives to be hedged. The basic principle may be in line with the general philosophy of financial engineering. Different delta hedging strategies have been compared in De Giovanni *et al.* (2008) and some standard hedging strategies in market have been discussed in Neugebauer *et al.* (2006). Our paper's main goal is to consider hedging basket CDSs by using their own underlying single-name CDSs as hedging instruments, which is an extension of the work in Zhu *et al.* (2014). We first discuss the pricing for both basket CDSs and single CDSs. After gaining some insights in these calculation methods, we then focus on the hedging of basket CDSs by using delta hedge (De Giovanni *et al.*, 2008) and delta-gamma hedge (Castellacci & Siclari, 2003). More precisely, we intend to hedge by considering the price movements in single-name CDSs with respect to the shifting in the correlation, which is one of the parameters of Gaussian copula model. The key advantage of our hedging method is that we use a smaller number of single-name CDSs as the delta hedging instruments. Furthermore, we also proposed the delta-gamma hedge method by using the underlying defaultable single-name CDSs as the hedging instruments, which not only ensures delta hedge but also guarantees gamma hedge. A hedging measure would be developed for testing and making comparison with the simulated hedging results.

The paper is structured as follows. Section 2 presents the method for computing the joint probability distribution of default times which is then applied to three models with different dependent structures. Section 3 considers the pricing of credit default swaps under each of the three models. Section 4 discusses the hedging strategies including the delta hedge and delta-gamma hedge that is applicable to each model and provides corresponding measures for testing the proposed hedging methods. We conduct numerical experiments on hedging basket CDSs for several situations in Section 5. Finally, Section 6 concludes the paper.

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2. Models for dependent defaults

In this section, we study three classes of models: the homogeneous Gaussian copula model with one common factor and individual correlations (Model I); the heterogeneous Gaussian copula model with different categories and single correlation, and there is one individual common factor in each category (Model II); the homogeneous Gaussian copula model with several common factors (Model III), where the number of correlations should be equal to the number of factors. Under different situations, corresponding models may be chosen accordingly. We shall present the details of these models below.

In what follows, we consider N defaultable claims, denoted as $1, 2, \ldots, N$. For each name i, the associated default time is denoted by $\tau_i, i = 1, \ldots, N$, and these default times are all defined on a given complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Our market model is supposed to be arbitrage-free, and $\mathbb{P}$ is a risk-neutral probability measure we adopted. Here, we suppose for simplicity that the risk-neutral probability measure is clearly given. The marginal distribution function of the default time τ_i is supposed to be $F_i(t) = \mathbb{P}(\tau_i \le t)$. Let $F(t_1, \ldots, t_N) = \mathbb{P}(\tau_1 \le t_1, \ldots, \tau_N \le t_N)$ be the joint cumulative probability distribution function of the N default times. By Sklar's theorem (Sklar, 1959), there exists a copula function $C(u_1, u_2, \ldots, u_N)$ such that

$$F(t_1, t_2, \ldots, t_N) = C(F_1(t_1), F_2(t_2), \ldots, F_N(t_N)). \tag{1}$$

For each $i = 1, 2, \ldots, N$, we denote the default indicator process $H_t^i = I_{\{\tau_i \le t\}}$ and the corresponding filtration $\mathbb{H}^i = (\mathcal{H}_t^i)_{t \in \mathbb{R}_+}$, where $\mathcal{H}_t^i$ is the $\mathbb{P}$-completion of the σ-algebra $\sigma(H_s^i : s \le t)$. In the succeeding part, we let $\mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2 \vee \cdots \vee \mathcal{H}_t^N$, where $\mathcal{H}_t$ is the minimal σ-algebra containing $\mathcal{H}_t^1, \mathcal{H}_t^2, \ldots, \mathcal{H}_t^N$. We assume that $\mathbb{P}(\tau_i = \tau_j) = 0$ for any $1 \le i \ne j \le N$. Then the collection of default times can be put in the order $\tau_1 < \cdots < \tau_i < \cdots < \tau_N$, where τ_i is the time of the i-th default.

2.1 *Homogeneous Gaussian copula model with one common factor (Model I)*

We consider a Gaussian vector $(X_1, X_2, \ldots, X_N)$, where $X_i = \rho_i Z + \sqrt{1 - \rho_i^2} Z_i$, $\rho_i \in (-1, 1)$ and $Z, Z_i (1 \le i \le N)$ are independent and identically distributed standard normal variables. Then

$$\operatorname{cov}(X_i, X_j) = \begin{cases} 1 & i = j \\ \rho_i \rho_j & i \ne j. \end{cases} \tag{2}$$

We shall consider Gaussian copula in the following part:

$$C(u_1, \ldots, u_N) = \Phi_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N)), \tag{3}$$

where Φ denotes the cumulative distribution function of the standard normal $N(0, 1)$, and Σ is the covariance matrix of X_i $(1 \le i \le N)$. Φ_Σ is the cumulative joint probability distribution function of a multivariate normal distribution with covariance matrix Σ.

Under the homogeneous Gaussian copula model, $\Phi(X_i) < F_i(t_i)$ implies $X_i < \Phi^{-1}(F_i(t_i))$, so

$$Z_i < \frac{\Phi^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1 - \rho_i^2}} \quad \text{and} \quad F_i(t_i|Z) = \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1 - \rho_i^2}}\right), \tag{4}$$

where $F_i(t_i|Z) = \mathbb{P}(\tau_i \leq t|Z)$. Since $Z_i, i = 1, 2, \ldots, N$ are independent, the joint cumulative distribution of default time given Z would be

$$F(t_1, \cdots, t_N|Z) = \prod_{i=1}^{N} F_i(t_i|Z) = \prod_{i=1}^{N} \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1-\rho_i^2}}\right) \tag{5}$$

and

$$C(F_1(t_1), \cdots, F_N(t_N)) = F(t_1, \cdots, t_N) = \int \left(\prod_{i=1}^{N} \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i x}{\sqrt{1-\rho_i^2}}\right)\right) \phi(x)\mathrm{d}x, \tag{6}$$

where $\phi(x)$ is the probability density function of Z.

2.2 *Heterogeneous Gaussian copula model with different categories (Model II)*

In this model, we consider a Gaussian vector $(X_1^j, X_2^j, \ldots, X_{N_j}^j)$ in category j and it has N_j defaultable claims in this category $(1 \leq j \leq J)$, where $X_i^j = \rho Z^j + \sqrt{1-\rho^2} Z_i^j$, $\rho \in (-1, 1)$, and Z^j, $Z_i^j (1 \leq j \leq J), i = 1, \ldots, N_j$ are independent and identically distributed standard normal variables. The J categories here may be interpreted as J different industrial sectors, and clearly

$$\mathrm{cov}(X_i^j, X_k^j) = \begin{cases} 1 & i = k \\ \rho^2 & i \neq k. \end{cases} \tag{7}$$

We denote the marginal distribution function of default time τ_i^j by $F_i^j(t) = \mathbb{P}(\tau_i^j \leq t)$ and the joint probability distribution function of default times by

$$F^j(t_1, \ldots, t_{N_j}) = \mathbb{P}(\tau_1^j \leq t_1, \ldots, \tau_{N_j}^j \leq t_{N_j}). \tag{8}$$

Similar to the previous case, the joint cumulative distribution function can be written as follows:

$$C^j(F_1^j(t_1), \cdots, F_{N_j}^j(t_{N_j})) = F^j(t_1, \cdots, t_{N_j}) = \int \left(\prod_{i=1}^{N_j} \Phi\left(\frac{\Phi^{-1}(F_i^j(t_i)) - \rho x}{\sqrt{1-\rho^2}}\right)\right) \phi(x)\mathrm{d}x. \tag{9}$$

For this case, notice that $\mathbb{P}(\tau_i^j = \tau_k^j) = 0$ for any $1 \leq i \neq k \leq N_j$, $\mathbb{P}(\tau_i^j = \tau_k^l) = 0$ for any $1 \leq i \leq N_j$, $1 \leq k \leq N_l, j, l = 1, \ldots J$.

2.3 *Homogeneous Gaussian copula model with several common factors (Model III)*

In Model III, we consider a Gaussian vector $(X_1, X_2, \ldots, X_N)$, where

$$X_i = \rho_1 Z^1 + \ldots + \rho_J Z^J + \sqrt{1 - \rho_1^2 - \ldots - \rho_J^2} Z_i. \tag{10}$$

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Suppose $\rho_j \in (-1, 1)$ are pairwise different, $j = 1, \ldots, J$ and $Z_i, i = 1, \ldots, N$ are independent and identically distributed standard normal variables. Then

$$\mathrm{cov}(X_i, X_j) = \begin{cases} 1 & i = j \\ \sum_{q=1}^{J} \rho_q^2 & i \neq j. \end{cases} \tag{11}$$

The joint cumulative probability distribution is given by:

$$\begin{aligned} C(F_1(t_1), \ldots, F_N(t_N)) &= F(t_1, \cdots, t_N) \\ &= \int \left(\prod_{i=1}^{N} \Phi \left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_1 x_1 - \ldots - \rho_J x_J}{\sqrt{1 - \rho_1^2 - \ldots - \rho_J^2}} \right) \right) p(x_1, \ldots, x_J) \mathrm{d}x_1, \cdots, \mathrm{d}x_J, \end{aligned} \tag{12}$$

where $p(x_1, \ldots, x_J)$ denotes the joint probability density function of $Z^1, \cdots, Z^J$ and

$$p(x_1, \cdots, x_J) = \frac{1}{(2\pi)^{\frac{J}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \sum_{m,n=1}^{J} r_{mn} x_m x_n \right\}, \tag{13}$$

where

$$\Sigma = \mathbb{E}[(Z^1, \cdots, Z^J)^T \cdot (Z^1, \cdots, Z^J)] \quad \text{and} \quad \Sigma^{-1} = (r_{mn}). \tag{14}$$

Here, we only consider the Gaussian copula models; however, we would like to point it out that the ideas behind the formulas for pricing and hedging derived in this paper are still valid for other copula models. That is to say, the pricing and hedging methods introduced in this paper are applicable to more general cases and similar formulas could be derived under different models with the same methods.

3. Pricing credit derivatives

In this section, we consider the pricing of the K^{th}-to-default swap on a set of N defaultable claims (see, for example, Giesecke *et al.*, 2011; Gu *et al.*, 2013, 2014) and the chosen survival single-name CDSs. Suppose premium payments of a basket CDS are made on dates $0 < t^1 < t^2 < \cdots < t^M = T$, where T is the expiry date. The periodic premium paid at each time slot is supposed to be deterministic Y, which is chosen such that the value of the CDS contract is equal to 0 at the issue time. We assume the default payments are made immediately after the default and the deterministic recovery rate is given by δ. For the single-name CDSs, the premiums payment date and the expiry date are supposed to be the same as in the basket default swap contract, which are $t^i (1 \leq i \leq M)$ and T, respectively. We denote the nominal of a given reference credit by 1 and the constant risk-free interest rate by $r > 0$.

Since the pricing methods under the three models are similar to each other, we will give more details of the results under Model I for illustration. However, results regarding to Models II and III will also be discussed and given. In our numerical experiments, different models will be taken into consideration. For the pricing of the K^{th}-to-default swap over N defaultable claims, we have:

PROPOSITION 1 The value of K^{th}-to-default CDS at time t under Model I is given in the following form:

$$
\begin{aligned}
V_K(t,\rho_1,\cdots,\rho_N) \;=\; & (1-\delta)\left[G^K(t,t)-G^K(T,t)e^{-r(T-t)}-r\int_t^T G^K(x,t)e^{-r(x-t)}\mathrm{d}x\right] \\
& -\sum\nolimits_{t^M\geq t^{j+1}>t^j>t}\Big[YG^K(t^{j+1},t)e^{-r(t^{j+1}-t)}-\tfrac{Y(1+rt^j)}{t^{j+1}-t^j} \\
& \int_{t^j}^{t^{j+1}} G^K(x,t)e^{-r(x-t)}\mathrm{d}x+\tfrac{Yr}{t^{j+1}-t^j}\int_{t^j}^{t^{j+1}} G^K(x,t)xe^{-r(x-t)}\mathrm{d}x\Big],
\end{aligned}
\tag{15}
$$

where $G^K(t,s)=\mathbb{P}(\tau_K>t|\mathcal{H}_s)$ and $s<t,\tau_K>s$, r is the given constant interest rate.

Proof. The proof is given in Appendix 7.1. □

COROLLARY 1 In particular, if the accrued premium payments due to the defaults between the premium payments dates is not taken into consideration, then the calculation of premium could be simplified. Therefore, the value of this K^{th}-to-default CDS at time t could be written in the following way:

$$
\begin{aligned}
V_K(t,\rho_1,\cdots,\rho_N) \;=\; & (1-\delta)\left[G^K(t,t)-G^K(T,t)e^{-r(T-t)}-r\int_t^T G^K(x,t)e^{-r(x-t)}\mathrm{d}x\right] \\
& -\sum\nolimits_{t^M>t^j>t} Ye^{-r(t^j-t)}G^K(t^j,t),
\end{aligned}
\tag{16}
$$

where $G^K(t,s)$ denotes the same as in Proposition 1 and $s<t,\tau_K>s$.

Considering the accrued premium payments will make the calculation more cumbersome, without loss of generality, in the following content, we will not consider the accrued premium payments due to the defaults between the premium payments dates. However, all the ideas, methods and formulas would remain the same except the expression of premium is different in $V_K(t,\rho_1,\cdots,\rho_N)$. Corresponding expression for the premium could be chosen according to the real situation.

REMARK 1 Under Models II and III, the pricing formula Equations (15) and (16) is still valid except under Model II, $G^K(t,s)$ should be changed to $G_j^K(t,s)=\mathbb{P}(\tau_K^j>t|\mathcal{H}_s^j)$ when the category j is considered.

To give $G^K(t,s)$, suppose at time $0<s\leq T$, k_s names out of a total of N names have already defaulted. For convenience, the surviving names are denoted as $\{j_1,j_2,\ldots,j_{N-k_s}\}$, and the k_s defaulted times of names $i_1,\ldots,i_{k_s}$ are ordered as $u_1<\cdots<u_{k_s}$. We write

$$
D_{k_s}=\{x_{i_1}=u_1,\ldots,x_{i_{k_s}}=u_{k_s}\}. \tag{17}
$$

In the succeeding part, we denote the k_s names that default before time s as names $\{i_1,\ldots,i_{k_s}\}$, the l names that default between time s and t as names $\{q_1,\ldots,q_l\}$ and the $N-k_s-l$ names that default after time t will be denoted as $\{m_1,\ldots,m_{N-k_s-l}\}$.

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PROPOSITION 2 For all $i = 1, 2, \ldots, N$, the distribution functions of default time $F_i(t)$ are distinct, and the correlation coefficients are distinct, under Model I, we have

$$G^K(t,s) = \frac{\sum_{l=0}^{K-k_s-1} \sum_{I^l} \int_{A_1} f(t_1,\ldots,t_N)|_{D_{k_s}} \mathrm{d}t_{j_1} \ldots \mathrm{d}t_{j_{N-k_s}}}{\int_{B_1} f(t_1,\ldots,t_N)|_{D_{k_s}} \mathrm{d}t_{j_1} \ldots \mathrm{d}t_{j_{N-k_s}}}, \tag{18}$$

where

$$A_1 := \underbrace{[s,t] \times \cdots \times [s,t]}_{x_{j_i} \in I^l} \underbrace{[t,+\infty] \times \cdots \times [t,+\infty]}_{N-k_s-l}, i = 1, \cdots, N-k_s \quad \text{and} \quad B_1 := \underbrace{[s,+\infty] \ldots [s,+\infty]}_{N-k_s},$$

$$f(t_1,\ldots,t_N) = \int \left(\prod_{i=1}^{N} \frac{\partial}{\partial t_i} \Phi \left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i x}{\sqrt{1-\rho_i^2}} \right) \right) \phi(x)\mathrm{d}x, \tag{19}$$

and $f(t_1,\ldots,t_N)|_{D_{k_s}}$ denotes $f(t_{i_1} = u_1, \cdots, t_{i_{k_s}} = u_{k_s}, t_{j_1}, \cdots, t_{j_{N-k_s}})$. Here, I^l denotes the set of all possible l elements chosen from $N - k_s$ surviving names.

Proof. The proof is given in Appendix 7.2. □

COROLLARY 2 Under Model II, we need to consider $G_j^K(t,s) = \mathbb{P}(\tau_K^j > t|\mathcal{H}_s^j)$ for each category j where $\mathcal{H}_t^j = \mathcal{H}_t^1 \vee \mathcal{H}_t^2 \vee \cdots \vee \mathcal{H}_t^{N_j}$. Proposition 2 is valid, and the formula for $G_j^K(t,s)$ is as same as Equation (18) except N needs to change to N_j and the $f(t_1,\ldots,t_N)$ need to be replaced by $f^j(t_1,\ldots,t_{N_j})$. Under this model,

$$f^j(t_1,\ldots,t_{N_j}) = \int \left(\prod_{i=1}^{N_j} \frac{\partial}{\partial t_i} \Phi \left(\frac{\Phi^{-1}(F_i^j(t_i)) - \rho x}{\sqrt{1-\rho^2}} \right) \right) \phi(x)\mathrm{d}x \tag{20}$$

and $f^j(t_1,\ldots,t_{N_j})|_{D_{k_s}} = f^j(t_{i_1} = u_1, \cdots, t_{i_{k_s}} = u_{k_s}, t_{l_1}, \cdots, t_{l_{N_j-k_s}})$.

COROLLARY 3 Under Model III, we also need to consider $G^K(t,s)$ and Proposition 2 is still valid except

$$f(t_1,\ldots,t_N) = \int \left(\prod_{i=1}^{N} \frac{\partial}{\partial t_i} \Phi \left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_1 x_1 - \cdots - \rho_J x_J}{\sqrt{1-\rho_1-\cdots-\rho_J}} \right) \right) p(x_1,\cdots,x_J)\mathrm{d}x_1,\cdots,\mathrm{d}x_J. \tag{21}$$

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PROPOSITION 3 If all the distribution functions of default times $F_i(t)(i = 1, \ldots, N)$ are identical, and $\rho_1 = \rho_2 = \cdots = \rho_N$, then under Model I,

$$G^K(t,s) = \frac{\sum_{l=0}^{K-k_s-1} \binom{N-k_s}{l} \int_{A_2} f(t_1,\ldots,t_N)|_{D_{k_s}} \mathrm{d}t_{j_1} \ldots \mathrm{d}t_{j_{N-k_s}}}{\int_{B_2} f(t_1,\ldots,t_N)|_{D_{k_s}} \mathrm{d}t_{j_1} \ldots \mathrm{d}t_{j_{N-k_s}}}, \tag{22}$$

where

$$A_2 := \underbrace{[s,t] \times \cdots \times [s,t]}_{l} \times \underbrace{[t,+\infty] \times \cdots \times [t,+\infty]}_{N-k_s-l}, \quad B_2 := \underbrace{[s,+\infty] \times \cdots \times [s,+\infty]}_{N-k_s},$$

and $f(t_1,\ldots,t_N)|_{D_{k_s}}$ denotes the same as in Proposition 2.

Proof. The proof is straightforward from the proof for Proposition 2 and therefore omitted here. □

REMARK 2 For Models II and III, Proposition 3 is still valid with the corresponding N_j, N and formulas for $f^j(t_1,\ldots,t_{N_j})|_{D_{k_s}}$ and $f(t_1,\ldots,t_N)|_{D_{k_s}}$ given in Corollaries 2 and 3.

The numerator of $G^K(t,s)$: $C = \int_{A_2} f(t_1,\ldots,t_N)|_{D_{k_s}} \mathrm{d}t_{j_1} \ldots \mathrm{d}t_{j_{N-k_s}}$ and the denominator of $G^K(t,s)$: $D = \int_{B_2} f(t_1,\ldots,t_N)|_{D_{k_s}} \mathrm{d}t_{j_1} \ldots \mathrm{d}t_{j_{N-k_s}}$ can be calculated as follows:

(I) Under Model I,
Case (i) $l = 0$, the numerator becomes

$$C = \int \prod_{j=1}^{k_s} \frac{\partial g_{i_j}(t_{i_j}, \rho_{i_j}, x)}{\partial t_{i_j}} \bigg|_{t_{i_j}=u_j} \prod_{j=1}^{N-k_s} (1 - g_{m_j}(t, \rho_{m_j}, x)) \phi(x) \mathrm{d}x. \tag{23}$$

Case (ii) $0 < l \leq N - k_s$, the numerator becomes

$$C = \int \prod_{j=1}^{k_s} \frac{\partial g_{i_j}(t_{i_j}, \rho_{i_j}, x)}{\partial t_{i_j}} \bigg|_{t_{i_j}=u_j} \prod_{j=1}^{l} (g_{q_j}(t, \rho_{q_j}, x) - g_{q_j}(s, \rho_{q_j}, x)) \prod_{j=1}^{N-k_s-l} (1 - g_{m_j}(t, \rho_{m_j}, x)) \phi(x) \mathrm{d}x. \tag{24}$$

The denominator $D = \int_{B_2} f(t_1,\ldots,t_N)|_{D_{k_s}} \mathrm{d}t_{j_1} \ldots \mathrm{d}t_{j_{N-k_s}}$ can be calculated as follows:

$$D = \int \prod_{j=1}^{k_s} \frac{\partial g_{i_j}(t_{i_j}, \rho_{i_j}, x)}{\partial t_{i_j}} \bigg|_{t_{i_j}=u_j} \prod_{j=1}^{N-k_s} (1 - g_{q_j}(s, \rho_{q_j}, x)) \phi(x) \mathrm{d}x, \tag{25}$$

where

$$g_j(t_i, \rho_j, x) = \Phi\left(\frac{\Phi^{-1}(F_j(t_i)) - \rho_j x}{\sqrt{1-\rho_j^2}}\right). \tag{26}$$

The numerator and denominator of $G_j^K(t,s)$ and $G^K(t,s)$ under Models II and III can be calculated very similarly as above. More details are given below.

(II) Under Model II, the calculation for C and D under Model I is still valid except $g_j(t_i,\rho_j,x)$ in Equation (26) should be replaced by

$$g_k^j(t_i,\rho,x) = \Phi\left(\frac{\Phi^{-1}(F_k^j(t_i)) - \rho x}{\sqrt{1-\rho^2}}\right) \tag{27}$$

for each category j ($j = 1,\cdots,J$).

(III) Under Model III, the calculation for C and D under Model I is still valid except $g_j(t_i,\rho_j,x)$ in Equation (26), $\phi(x)$ and $\mathrm{d}x$ in the formulas should be replaced by

$$g_j(t_i,\rho,x) = g_j(t_i,\rho_1,\cdots,\rho_J,x) = \Phi\left(\frac{\Phi^{-1}(F_j(t_i)) - \rho_1 x_1 - \cdots - \rho_J x_J}{\sqrt{1-\rho_1-\cdots-\rho_J}}\right), \tag{28}$$

$p(x_1,\cdots,x_J)$ and $\mathrm{d}x_1,\cdots,\mathrm{d}x_J$, respectively.

Without loss of generality, we assume that the chosen single-name CDS whose underlying asset is name i, which is one of the original N defaultable names, and the corresponding recovery rate is δ^i, the periodic paid premium is Y^i. The value of single-name-i CDS on a set of N defaultable claims at time t is

$$\begin{aligned}\bar{V}_i(t,\rho_1,\ldots,\rho_N) &= (1-\delta^i)\left[\bar{G}_i(t,t) - \bar{G}_i(T,t)e^{-r(T-t)} - r\int_t^T \bar{G}_i(x,t)e^{-r(x-t)}\mathrm{d}x\right] \\ &\quad - \sum_{t^M>t^j>t} Y^i e^{-r(t^j-t)}\bar{G}_i(t^j,t),\end{aligned} \tag{29}$$

where $\bar{G}_i(t,s) = \mathbb{P}(\tau_i > t|\mathcal{H}_s)$ and $s < t$, $\tau_i > s$, r is the given constant interest rate.

To give $\bar{G}_i(t,s)$, we have the same notations as before, and we still suppose at time $0 < s \le T$, k_s names out of a total of N names have already defaulted. Let D_{k_s} denote the same as Equation (17), then the survival function $\bar{G}_i(t,s)$ is given as follows:

$$\bar{G}_i(t,s) = \frac{\displaystyle\int_{A_3} f(t_1,\ldots,t_N)|_{D_{k_s}}\mathrm{d}t_{j_1}\ldots\mathrm{d}t_{j_{N-k_s}}}{\displaystyle\int_{B_3} f(t_1,\ldots,t_N)|_{D_{k_s}}\mathrm{d}t_{j_1}\ldots\mathrm{d}t_{j_{N-k_s}}}, \tag{30}$$

where

$$A_3 = [t,+\infty]\times\underbrace{[s,+\infty]\times\cdots\times[s,+\infty]}_{N-k_s-1},\quad B_3 = \underbrace{[s,+\infty]\times\cdots\times[s,+\infty]}_{N-k_s}$$

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and

(I) under Model I, $f(t_1,\dots,t_N)$ denotes Equation (19),

(II) under Model II, N needs to change to N_j, $f^j(t_1,\dots,t_{N_j})$ need to be considered as demonstrated in Corollary 1 and it denotes Equation (20),

(III) under Model III, $f(t_1,\dots,t_N)$ denotes Equation (21).

Here, $f(t_1,\dots,t_N)|_{D_{k_s}}$ and $f^j(t_1,\dots,t_{N_j})|_{D_{k_s}}$ denotes the same thing as given in Proposition 2 and Corollary 2.

Under Model I, the numerator of $\bar{G}_i(t,s)$: $\bar{C}=\int_{A_3} f(t_1,\dots,t_N)_{D_{k_s}}\mathrm{d}t_{j_1}\dots\mathrm{d}t_{j_{N-k_s}}$ is

$$\bar{C}=\int\prod_{j=1}^{k_s}\frac{\partial g_{i_j}(t_{i_j},\rho_{i_j},x)}{\partial t_{i_j}}\bigg|_{t_{i_j}=u_j}\prod_{j=1}^{N-k_s-1}(1-g_{q_j}(s,\rho_{q_j},x))(1-g_i(t,\rho_i,x))\phi(x)\mathrm{d}x \tag{31}$$

and the denominator of $\bar{G}_i(t,s)$: $\bar{D}=\int_{B_3} f(t_1,\dots,t_N)|_{D_{k_s}}\mathrm{d}t_{j_1}\dots\mathrm{d}t_{j_{N-k_s}}$ can be written as follows:

$$\bar{D}=\int\prod_{j=1}^{k_s}\frac{\partial g_{i_j}(t_{i_j},\rho_{i_j},x)}{\partial t_{i_j}}\bigg|_{t_{i_j}=u_j}\prod_{j=1}^{N-k_s}(1-g_{q_j}(s,\rho_{q_j},x))\phi(x)\mathrm{d}x, \tag{32}$$

where $g_j(t_i,\rho_j,x)$ is the same as Equation (26). The numerator and denominator of $\bar{G}_i^j(t,s)$ and $\bar{G}_i(t,s)$ under Models II and III can be calculated very similarly according to formulas given by Equations (31) and (32) with Equations (27) and (28).

4. Hedging shifts in the correlation

In this section, we shall discuss the hedging of basket CDSs by some underlying single-name CDSs with respect to correlation parameters. If any of the selected hedging instruments defaulted, one could choose new instruments from the remaining survival single-name CDSs. Both of delta and delta-gamma hedging are considered. In this section, we give the hedging strategies that are applicable to general cases. For different models, the corresponding valuation formulas and correlations ρ or $\rho_1,\cdots,\rho_n (n=N,J)$ in the hedging strategies should be chosen accordingly. In order to illustrate it clearly and avoid confusion, similar to the discussion of pricing, we also use Model I as an example to give the illustration. Results regarding to Models II and III will also be discussed and given.

For Model I, we first consider a single-name CDS of name i $(1\le i\le N)$ that is randomly chosen from the set of N defaultable claims. The approaches that we adopted for hedging basket CDSs includes delta and delta-gamma hedging with respect to shifting in the correlation coefficients $\rho_1,\dots,\rho_N$, while other parameters remain unchanged. For simplicity of discussion, we denote

$$V_K(t,\boldsymbol{\rho})=V_K(t,\rho_1,\dots,\rho_N),\quad \phi^i(t,\boldsymbol{\rho})=\phi^i(t,\rho_1,\dots,\rho_N) \tag{33}$$

and

$$\bar{V}_i(t,\boldsymbol{\rho})=\bar{V}_i(t,\rho_1,\dots,\rho_N). \tag{34}$$

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Then we have the following proposition.

PROPOSITION 4 Suppose that $\rho_{n_1}, \cdots, \rho_{n_q}$ are pairwise different and they are selected from $\rho_1, \cdots, \rho_N$ under Model I, then we require q single-name CDSs: $i_1, \cdots, i_q$ as the hedging instruments to achieve the goal of delta hedge. Let

$$\Phi(t, \boldsymbol{\rho}) = (\phi^{i_1}(t, \boldsymbol{\rho}), \ldots, \phi^{i_q}(t, \boldsymbol{\rho}))^T, \tag{35}$$

where $\phi^{i_j}(t, \boldsymbol{\rho})$ is the corresponding hedging position for the single-name-i_j CDS. When $H_q(t, \boldsymbol{\rho})$ is invertible, that is to say $\det|H_q(t, \boldsymbol{\rho})|$ is non-zero holds for all t we concerned, then we can get the unique expression of $\Phi(t, \boldsymbol{\rho})$ from

$$\Phi(t, \boldsymbol{\rho}) = H_q^{-1}(t, \boldsymbol{\rho}) \cdot \left(\frac{\partial}{\partial \rho_{n_1}} V_K(t, \boldsymbol{\rho}), \ldots, \frac{\partial}{\partial \rho_{n_q}} V_K(t, \boldsymbol{\rho}) \right)^T, \tag{36}$$

where

$$H_q(t, \boldsymbol{\rho}) = \begin{pmatrix} \frac{\partial}{\partial \rho_{n_1}} \bar{V}_{i_1}(t, \boldsymbol{\rho}) & \cdots & \frac{\partial}{\partial \rho_{n_1}} \bar{V}_{i_q}(t, \boldsymbol{\rho}) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \rho_{n_q}} \bar{V}_{i_1}(t, \boldsymbol{\rho}) & \cdots & \frac{\partial}{\partial \rho_{n_q}} \bar{V}_{i_q}(t, \boldsymbol{\rho}) \end{pmatrix}. \tag{37}$$

Proof. The proof is given in Appendix 7.3. □

REMARK 3 If $\rho_1 = \rho_2 = \cdots = \rho_N = \rho$, we only need one hedge instrument, and

$$\phi^i(t, \rho_1, \cdots, \rho_N) = \phi^i(t, \rho) = \frac{dV_K(t, \rho)}{d\bar{V}_i(t, \rho)} = \frac{\frac{\partial}{\partial \rho} V_K(t, \rho)}{\frac{\partial}{\partial \rho} \bar{V}_i(t, \rho)}, \tag{38}$$

where the single-name-i CDS is chosen as the hedge instrument.

COROLLARY 4 Under Model II, Proposition 4 is valid for delta hedge. There is only one correlation: ρ, we need one single-name CDS as the hedging instrument to achieve the goal of delta hedge. The delta hedge strategy is given by Equation (38). To be more specifically, to hedge the K^{th}-to-default basket CDS in category j, the hedging position in single-name CDS i in category l is equal to

$$\phi^i_{j,l}(t, \rho) = \frac{dV^j_K(t, \rho)}{d\bar{V}^l_i(t, \rho)} = \frac{\frac{\partial}{\partial \rho} V^j_K(t, \rho)}{\frac{\partial}{\partial \rho} \bar{V}^l_i(t, \rho)}. \tag{39}$$

REMARK 4 If all $\rho_1, \rho_2, \cdots, \rho_N$ are distinct, then the number of hedge instruments should be N, so all the single name CDSs should be used as hedging instruments. We can get

$$\Phi(t, \rho_1, \cdots, \rho_N) = H^{-1}(t, \rho_1, \cdots, \rho_N) \cdot \left(\frac{\partial}{\partial \rho_1} V_K(t, \rho_1, \cdots, \rho_N), \cdots, \frac{\partial}{\partial \rho_N} V_K(t, \rho_1, \cdots, \rho_N) \right)^T \tag{40}$$

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if $H(t,\rho_1,\cdots,\rho_N)$ is invertible, where

$$H(t,\rho_1,\cdots,\rho_N)=\begin{pmatrix}\frac{\partial}{\partial\rho_1}\bar{V}_1(t,\rho_1,\cdots,\rho_N) & \cdots & \frac{\partial}{\partial\rho_1}\bar{V}_N(t,\rho_1,\cdots,\rho_N)\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial\rho_N}\bar{V}_1(t,\rho_1,\cdots,\rho_N) & \cdots & \frac{\partial}{\partial\rho_N}\bar{V}_N(t,\rho_1,\cdots,\rho_N)\end{pmatrix}. \tag{41}$$

COROLLARY 5 Under Model III, Proposition 4 is still valid for delta hedge. There are J distinct correlations: $\rho_1,\cdots,\rho_J$, we need J single-name CDSs: $i_1,\cdots,i_J$ as the hedging instruments to achieve the goal of delta hedge. More specifically, the delta hedge strategy is given by Equation (40) (with N change to J) if Equation (41) (N change to J) is invertible.

If all the default times of the chosen hedging instruments $\tau_i >= \tau_K$, $i=i_1,\ldots,i_q$, then the succeeding trading strategy $\Phi(t,\boldsymbol{\rho})$ can be applied to hedge the K^{th}-to-default basket CDS; otherwise, other assets would be chosen from the surviving assets to replace the defaulted ones. Then the corresponding trading strategy can be obtained.

As in Teixeira (2007), we shall measure the effectiveness of delta hedge by measuring the difference between the accumulated change in the values of the basket CDSs and the accumulated change in the value of the chosen single-name CDSs when correlation $\rho_{n_1},\ldots,\rho_{n_q}$ changes with time. Here, we use $\boldsymbol{\rho}$ to represent $(\rho_{i_1},\ldots,\rho_{i_q})$, suppose we consider the following portfolio:

$$\mathcal{V}_t=V_K(t,\rho_{i_1},\cdots,\rho_{i_q})-\sum_{h=1}^{q}\phi^{i_h}(t,\rho_{i_1},\cdots,\rho_{i_q})\bar{V}_{i_h}(t,\rho_{i_1},\cdots,\rho_{i_q}) \tag{42}$$

and divide the interval $[t,\tau_K\wedge T]$ into J arbitrary sub-intervals such that $t=t_0\le t_{\le}\cdots\le t_{J-1}\le t_J=\tau_K\wedge T$. Here, we do not have any constraints on the sub-intervals, but for simplicity, we consider the identical sub-intervals in the numerical experiments. Suppose single-name CDSs: names $i_1,\ldots,i_q$ are chosen as the hedging instruments, then the following equation holds if the basket CDS is perfectly delta hedged:

$$\begin{aligned}V_K(t_J,\boldsymbol{\rho})-e^{r(t_J-t)}V_K(t,\boldsymbol{\rho}) \;=\; &\sum_{h=1}^{q}\sum_{j=1}^{J-1}e^{r(t_J-t_j)}\bar{V}_{i_h}(t_j,\boldsymbol{\rho})(\phi^{i_h}(t_{j+1},\boldsymbol{\rho})-2\phi^{i_h}(t_j,\boldsymbol{\rho})+\phi^{i_h}(t_{j-1},\boldsymbol{\rho}))\\ &+\textstyle\sum_{h=1}^{q}e^{r(t_J-t_0)}\bar{V}_{i_h}(t_0,\boldsymbol{\rho})(\phi^{i_h}(t_1,\boldsymbol{\rho})-2\phi^{i_h}(t_0,\boldsymbol{\rho}))\\ &+\sum_{h=1}^{q}\bar{V}_{i_h}(t_J,\boldsymbol{\rho})\phi^{i_h}(t_{J-1},\boldsymbol{\rho}),\end{aligned} \tag{43}$$

where r denotes the constant interest rate.

We may say that the hedge effect is better if the left-hand side of the above equation is closer to the right-hand side of this equation, then the combination of single-name-$i_1,\cdots,i_q$ CDSs that we take to hedge this K^{th}-to-default CDS is better. To further extend our results, we consider the delta-gamma hedge.

PROPOSITION 5 Assume that $\rho_{n_1},\cdots,\rho_{n_q}$ are different and they are selected from the set $\{\rho_1,\cdots,\rho_N\}$, we need $s=2q+\frac{q(q-1)}{2}$ single-name CDSs as hedge instruments, and we denote them as single-name-

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i_1, single-name-i_2, $\cdots$, single-name-i_s for uses in the delta-gamma hedge. Let

$$\bar{\Phi}(t,\boldsymbol{\rho}) = (\bar{\phi}_{i_1}(t,\boldsymbol{\rho}), \cdots, \bar{\phi}_{i_s}(t,\boldsymbol{\rho}))^T, \tag{44}$$

where $\bar{\phi}^{i_h}(t,\boldsymbol{\rho})$ is the corresponding hedging position for the single-name-i_h CDS, here $h = 1, \cdots, s$. If $\bar{H}_s(t,\boldsymbol{\rho})$ is invertible, that is to say det $|\bar{H}_s(t,\boldsymbol{\rho})|$ is non-zero holds for all t we concerned, then

$$\bar{\Phi}(t,\boldsymbol{\rho}) = \bar{H}_s^{-1}(t,\boldsymbol{\rho}) \cdot \left(\frac{\partial V_K}{\partial \rho_{n_1}}, \cdots, \frac{\partial V_K}{\partial \rho_{n_q}}, \frac{\partial^2 V_K}{\partial \rho_{n_1}^2}, \cdots, \frac{\partial^2 V_K}{\partial \rho_{n_q}^2}, 2\frac{\partial^2 V_K}{\partial \rho_{n_1}\partial \rho_{n_2}} \cdots, 2\frac{\partial^2 V_K}{\partial \rho_{n_i}\partial \rho_{n_j}}, \cdots 2\frac{\partial^2 V_K}{\partial \rho_{n_{q-1}}\partial \rho_{n_q}} \right)^T, \tag{45}$$

where

$$\bar{H}_s(t,\boldsymbol{\rho}) = \begin{pmatrix} \frac{\partial}{\partial \rho_{n_1}} \bar{V}_{i_1}(t,\boldsymbol{\rho}) & \cdots & \frac{\partial}{\partial \rho_{n_1}} \bar{V}_{i_s}(t,\boldsymbol{\rho}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial \rho_{n_q}} \bar{V}_{i_1}(t,\boldsymbol{\rho}) & \cdots & \frac{\partial}{\partial \rho_{n_q}} \bar{V}_{i_s}(t,\boldsymbol{\rho}) \\ \frac{\partial^2}{\partial \rho_{n_1}^2} \bar{V}_{i_1}(t,\boldsymbol{\rho}) & \cdots & \frac{\partial^2}{\partial \rho_{n_1}^2} \bar{V}_{i_s}(t,\boldsymbol{\rho}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial \rho_{n_q}^2} \bar{V}_{i_1}(t,\boldsymbol{\rho}) & \cdots & \frac{\partial^2}{\partial \rho_{n_q}^2} \bar{V}_{i_s}(t,\boldsymbol{\rho}) \\ 2\frac{\partial^2}{\partial \rho_{n_1}\partial \rho_{n_2}} \bar{V}_{i_1}(t,\boldsymbol{\rho}) & \cdots & 2\frac{\partial^2}{\partial \rho_{n_1}\partial \rho_{n_2}} \bar{V}_{i_s}(t,\boldsymbol{\rho}) \\ \vdots & & \vdots \\ 2\frac{\partial^2}{\partial \rho_{n_i}\partial \rho_{n_j}} \bar{V}_{i_1}(t,\boldsymbol{\rho}) & \cdots & 2\frac{\partial^2}{\partial \rho_{n_i}\partial \rho_{n_j}} \bar{V}_{i_s}(t,\boldsymbol{\rho}) \\ \vdots & & \vdots \\ 2\frac{\partial^2}{\partial \rho_{n_{q-1}}\partial \rho_{n_q}} \bar{V}_{i_1}(t,\boldsymbol{\rho}) & \cdots & 2\frac{\partial^2}{\partial \rho_{n_{q-1}}\partial \rho_{n_q}} \bar{V}_{i_s}(t,\boldsymbol{\rho}) \end{pmatrix}. \tag{46}$$

Proof. The proof can be found in Appendix 7.4. □

Note that only when $s \leq N$ the delta-gamma hedge would be possible, otherwise, we will not have enough hedging instruments for the problem.

Here, we consider the simplest situation that $\rho_1 = \rho_2 = \cdots = \rho_N = \rho$.

$$\mathcal{V}_t = V_K(t,\rho) - \bar{\phi}_i(t,\rho)\bar{V}_i(t,\rho) - \bar{\phi}_j(t,\rho)\bar{V}_j(t,\rho). \tag{47}$$

If this portfolio satisfies the conditions for the delta-gamma hedge, then the following two equations should be satisfied:

$$\begin{cases} \frac{\partial}{\partial \rho} V_K(t,\rho) &= \bar{\phi}_i(t,\rho)\frac{\partial}{\partial \rho}\bar{V}_i(t,\rho) + \bar{\phi}_j(t,\rho)\frac{\partial}{\partial \rho}\bar{V}_j(t,\rho) \\ \frac{\partial^2}{\partial \rho^2} V_K(t,\rho) &= \bar{\phi}_i(t,\rho)\frac{\partial^2}{\partial \rho^2}\bar{V}_i(t,\rho) + \bar{\phi}_j(t,\rho)\frac{\partial^2}{\partial \rho^2}\bar{V}_j(t,\rho) \end{cases} \tag{48}$$

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i.e.

$$\begin{pmatrix} \frac{\partial}{\partial\rho} V_K(t,\rho) \\ \frac{\partial^2}{\partial\rho^2} V_K(t,\rho) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial\rho}\bar{V}_i(t,\rho) & \frac{\partial}{\partial\rho}\bar{V}_j(t,\rho) \\ \frac{\partial^2}{\partial\rho^2}\bar{V}_i(t,\rho) & \frac{\partial^2}{\partial\rho^2}\bar{V}_j(t,\rho) \end{pmatrix} \begin{pmatrix} \bar{\phi}_i(t,\rho) \\ \bar{\phi}_j(t,\rho) \end{pmatrix}. \tag{49}$$

In order to ensure that a unique combination $\bar{\phi}_i(t,\rho)$ and $\bar{\phi}_j(t,\rho)$ exists, the following condition is required.

$$\det \begin{vmatrix} \frac{\partial}{\partial\rho}\bar{V}_i(t,\rho) & \frac{\partial}{\partial\rho}\bar{V}_j(t,\rho) \\ \frac{\partial^2}{\partial\rho^2}\bar{V}_i(t,\rho) & \frac{\partial^2}{\partial\rho^2}\bar{V}_j(t,\rho) \end{vmatrix} = \det |A_{ij}(t,\rho)| \neq 0. \tag{50}$$

There exists a unique solution $\bar{\phi}_i(t,\rho)$ and $\bar{\phi}_j(t,\rho)$ if and only if this matrix is invertible and

$$\begin{pmatrix} \bar{\phi}_i(t,\rho) \\ \bar{\phi}_j(t,\rho) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial\rho}\bar{V}_i(t,\rho) & \frac{\partial}{\partial\rho}\bar{V}_j(t,\rho) \\ \frac{\partial^2}{\partial\rho^2}\bar{V}_i(t,\rho) & \frac{\partial^2}{\partial\rho^2}\bar{V}_j(t,\rho) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial\rho} V_K(t,\rho) \\ \frac{\partial^2}{\partial\rho^2} V_K(t,\rho) \end{pmatrix}. \tag{51}$$

Then the corresponding single-name-i CDS and single-name-j CDS may be combined as a portfolio to be used as delta-gamma hedge instrument. Checking the existence of $\bar{\phi}_i(t,\rho)$ and $\bar{\phi}_j(t,\rho)$ would be necessary for delta-gamma hedge.

REMARK 5 If $\rho_1 = \rho_2 = \cdots = \rho_N = \rho$, then we need two single-name CDSs: name i and name j for the delta-gamma hedge. If $\bar{H}_2(t,\rho)$ is invertible, then we have the unique expressions for the corresponding hedging position of name i and j from

$$\begin{pmatrix} \bar{\phi}_i(t,\rho) \\ \bar{\phi}_j(t,\rho) \end{pmatrix} = \bar{H}_2(t,\rho)^{-1} \begin{pmatrix} \frac{\partial}{\partial\rho} V_K(t,\rho) \\ \frac{\partial^2}{\partial\rho^2} V_K(t,\rho) \end{pmatrix}, \tag{52}$$

where

$$\bar{H}_2(t,\rho) = \begin{pmatrix} \frac{\partial}{\partial\rho}\bar{V}_i(t,\rho) & \frac{\partial}{\partial\rho}\bar{V}_j(t,\rho) \\ \frac{\partial^2}{\partial\rho^2}\bar{V}_i(t,\rho) & \frac{\partial^2}{\partial\rho^2}\bar{V}_j(t,\rho) \end{pmatrix}. \tag{53}$$

COROLLARY 6 Under Model II, Proposition 5 is valid for delta-gamma hedge. There is only one correlation: ρ, we need two single-name CDSs as the hedging instruments to achieve the goal of delta-gamma hedge. The delta-gamma hedge strategy is given by Equation (52) with the chosen K^{th}-to-default CDS and portfolio of single-name CDSs.

REMARK 6 If all the $\rho_1, \ldots, \rho_N$ are distinct, then the number of parameters that needs to be hedged is $2N + \frac{N(N-1)}{2} > N$, where N is the number of underlying defaultable claims. This means the single-name CDSs are not sufficient for delta-gamma hedge in this situation.

COROLLARY 7 Under Model III, Proposition 5 is still valid for delta-gamma hedge. There are J distinct correlations: $\rho_1, \cdots, \rho_J$, we need $s = 2J + \frac{J(J-1)}{2}$ single-name CDSs: $i_1, \cdots, i_s$ as hedge instruments to achieve the goal of delta-gamma hedge. If $s > N$, the delta-gamma hedge becomes impossible because the number of single-name CDSs are not enough. If $s \leq N$, delta-gamma hedge becomes possible, and the delta-gamma hedge strategy is given by Equation (45) where $\rho_{n_1}, \cdots, \rho_{n_q}$ need to be changed to $\rho_1, \cdots, \rho_J$.

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REMARK 7 If there are $q(q \geq 1)$ different $\rho_{n_1}, \cdots, \rho_{n_q}$ chosen from the set $\{\rho_1, \cdots, \rho_N\}$, to make sure there are enough delta-gamma hedge instruments, q need to satisfy the following condition:

$$\frac{-3-\sqrt{9+8N}}{2} \leq q \leq \frac{-3+\sqrt{9+8N}}{2}, \tag{54}$$

since $q \geq 1$ and $\frac{-3-\sqrt{9+8N}}{2} \leq 0$, therefore

$$1 \leq q \leq \frac{-3+\sqrt{9+8N}}{2}, \tag{55}$$

otherwise delta-gamma hedge would become impossible.

The interval $[t, \tau_K \wedge T]$ is divided into J subintervals such that $t = t_0 \leq t_1 \leq \cdots \leq t_{J-1} \leq t_J = \tau_K \wedge T$. The following equation holds if the basket CDS is perfectly delta-gamma hedged by some portfolio consisting of single-name-i_h CDS, $h = 1, 2, \ldots, s$:

$$\begin{aligned}
&V_K(t_J, \boldsymbol{\rho}) - e^{r(t_J - t)} V_K(t, \boldsymbol{\rho}) \\
&= \sum_{h=1}^{s} \sum_{k=1}^{J-1} e^{r(t_J - t_k)} \bar{V}_{i_h}(t_k, \boldsymbol{\rho}) (\bar{\phi}_{i_h}(t_{k+1}, \boldsymbol{\rho}) - 2\bar{\phi}_{i_h}(t_k, \boldsymbol{\rho}) + \bar{\phi}_{i_h}(t_{k-1}, \boldsymbol{\rho})) \\
&\quad + \sum_{h=1}^{s} e^{r(t_J - t_0)} \bar{V}_{i_h}(t_0, \boldsymbol{\rho}) (\bar{\phi}^i(t_1, \boldsymbol{\rho}) - 2\bar{\phi}^i(t_0, \boldsymbol{\rho})) + \sum_{h=1}^{s} \bar{V}_{i_h}(t_J, \boldsymbol{\rho}) \bar{\phi}_{i_h}(t_{J-1}, \boldsymbol{\rho}),
\end{aligned} \tag{56}$$

where $\boldsymbol{\rho}$ represents $(\rho_{n_1}, \ldots, \rho_{n_q})$ and r denotes the constant interest rate. Similarly, this equality indicates that the hedge effect may be testified by comparison of the quantity of both sides. After adopting this discretization, one can simply check the value of det $|H(t_k, \boldsymbol{\rho})|$ for all $k = 1, \ldots, J-1$. If all of them are non-zero, then they can be employed. Consequently, the single-name-i_h CDS ($h = 1, \ldots, s$) could be combined to form a delta-gamma hedge portfolio.

5. Numerical experiments

In this section, some simulated numerical examples are presented to illustrate the prosed hedging method under various models. Without loss of generality, for simplicity, we shall consider the hedging of a first-to-default basket CDS. First, we give some hypothetical settings for the model parameters. We assume that all the underlying portfolios in different categories and different models consist of $N = 5$ defaultable claims. The marginal distribution function of default time τ_i is $F_i(t) = \mathbb{P}(\tau_i \leq t) = 1 - e^{-\lambda_i t}$ where λ_i is the corresponding intensity. We also suppose that the initial time $t_0 = 0$, the expiry date $T = 2$ years, the first default time is 1.5 years. The constant interest rate $r = 5\%$, and the premium will be paid quarterly. Here, regular and deterministic time steps are assumed. In particular, the discrete time step is assumed to be $\Delta t = t_j - t_{j-1} = \frac{1}{10}$ year.

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TABLE 1 *Hedge using only one single CDS in the same category*

Hedging instruments	Name 1	Name 2	Name 3
E	0.3208	0.1895	0.1140

5.1 *Hedging with one single-name CDS*

In this section, we discuss all the situations of hedging with only one single-name CDS. We consider Model I that there is only one category with the assumption that

$$\rho_1 = \rho_2 = \cdots = \rho_N = \rho = 0.6. \tag{57}$$

Let $a(t) = V_K(t_J, \rho) - e^{r(t_J - t)} V_K(t, \rho)$ and

$$\begin{aligned} b(t) \;=\; & -\textstyle\sum_{j=1}^{J-1} e^{r(t_J - t_j)} \bar{V}_i(t_j, \rho)(\phi^i(t_{j+1}, \rho) - 2\phi^i(t_j, \rho) + \phi^i(t_{j-1}, \rho)) \\ & - e^{r(t_J - t_0)} \bar{V}_i(t_0, \rho)(\phi^i(t_1, \rho) - 2\phi^i(t_0, \rho)) - \bar{V}_i(t_J, \rho)\phi^i(t_{J-1}, \rho). \end{aligned} \tag{58}$$

As introduced before, the degree of 'symmetry' of $a(t)$ and $b(t)$ would imply the hedge effect. The pricing functions here are evaluated by calculating the sum over all the discrete time points, and these time points are different from the previous hedging times (t_j) we used. Additionally, the following quantity is calculated:

$$E = \frac{1}{N} \sum_{k=0}^{N} [a(t_k) + b(t_k)]^2, \tag{59}$$

where $t_0 = 0$, $t_N = T \wedge \tau_K$ and τ_K is the default time of the K^{th}-to-default CDS. As the relation between $a(t)$ and $b(t)$, smaller of this quantity would be better. Three different underlying single-name CDSs are used as the hedging instruments to hedge one basket CDS. Suppose all the intensities are identical $\lambda = 0.5$, the recovery rate of the basket CDS is $\delta = 0.1$ and the recovery rates of three single-name CDSs: Name 1, Name 2 and Name 3 are $\delta^1 = 0.1$, $\delta^2 = 0.2$ and $\delta^3 = 0.15$, respectively. By shifting the correlation ρ, the dynamics of $a(t)$ and $b(t)$ are shown in Fig. 1. In addition, the corresponding results of E is given in Table 1.

Figure 1 shows that by shifting the correlation ρ, the accumulated change of the value in the basket CDS is approximately mirrored by the accumulated change of the value in the chosen hedging instruments. This indicates that the hedging method may be employed to reduce the risk attributed to fluctuations in the correlation ρ. We also note that the hedging results will be better if finer discretization time step is adopted for evaluating those pricing functions. Additionally, from Fig. 1 and Table 1, among all the three choices, if Name 3 is chosen as the hedging instrument, the hedging result would be the best.

Then we consider Model II that has different categories, and in each category, there are also several different single-name CDSs. The basket CDS that intend to be hedged is assumed to be in the first category. In the following numerical experiments, we take several different single-name CDSs in three different categories as the hedging instruments to illustrate our hedging strategy. Suppose the intensity in Category 1 is $\lambda^1 = 0.5$, in Category 2 is $\lambda^2 = 0.6$ and in Category 3 is $\lambda^3 = 0.4$. The recovery rate of the basket CDS in Category 1 is $\delta_1 = 0.1$, and the recovery rates of three single-name CDSs in this Category: Name 1, Name 2 and Name 3 are $\delta_1^1 = 0.1$, $\delta_1^2 = 0.2$ and $\delta_1^3 = 0.15$, respectively. Similarly,

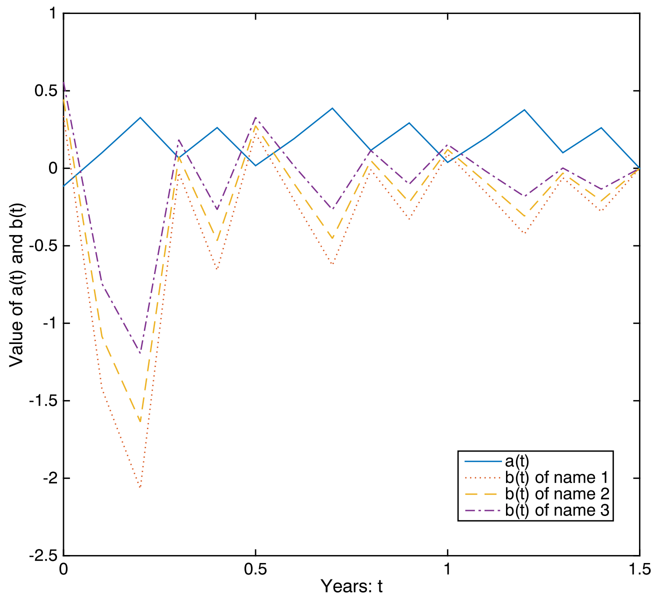

FIG. 1. Hedge using only one single CDS in the same category.

the recovery rates of two single-name CDSs in Category 2: Name 1 and Name 2 are $\delta_2^1 = 0.1$ and $\delta_2^2 = 0.2$, the recovery rates of two single-name CDSs in Category 3: Name 1 and Name 2 are $\delta_3^1 = 0.1$ and $\delta_3^2 = 0.2$, respectively. Therefore, let

$$a(t) = V_K^j(t_Q, \rho) - e^{r(t_Q - t)} V_K^j(t, \rho) \tag{60}$$

and

$$\begin{aligned} b(t) \;=\; & -\sum_{q=1}^{Q-1} e^{r(t_Q - t_q)} \bar{V}_i^l(t_q, \rho)(\phi_{j,l}^i(t_{q+1}, \rho) - 2\phi_{j,l}^i(t_q, \rho) + \phi_{j,l}^i(t_{q-1}, \rho)) \\ & - e^{r(t_Q - t_0)} \bar{V}_i^l(t_0, \rho)(\phi_{j,l}^i(t_1, \rho) - 2\phi_{j,l}^i(t_0, \rho)) - \bar{V}_i^l(t_Q, \rho)\phi_{j,l}^i(t_{Q-1}, \rho). \end{aligned} \tag{61}$$

We then repeat the same procedures in Model I for Model II and obtain the figures of $a(t)$ and $b(t)$s with shifting of the correlation $\rho = 0.6$. From Fig. 2, we notice that the performances of several concerned hedges are similar, that is to say, different hedging instruments may be employed. Quantitatively, we also have Table 2 below.

TABLE 2 *Hedge using only one single CDS in different categories*

Hedging instruments	Category 1			Category 2		Category 3	
	Name 1	Name 2	Name 3	Name 1	Name 2	Name 1	Name 2
E	0.3208	0.1140	0.1895	0.5965	0.2536	0.1624	0.1053

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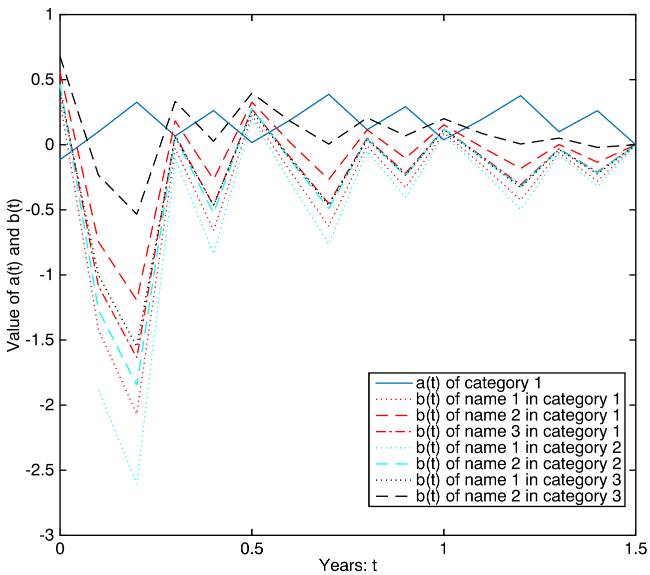

FIG. 2. Hedge using only one single CDS in different categories.

TABLE 3 *Hedge using two single CDSs*

Hedging instruments	Name 1 and Name 2	Name 1 and Name 3	Name 3 and Name 4
E	0.8802	0.5746	0.1242

5.2 *Hedging with several single-name CDSs*

In this section, we shall discuss the case where several underlying single-name CDSs are chosen as the hedging instruments so as to hedge against changes in correlations in the basket CDSs. Without loss of generality, we take delta hedge of the fluctuation of the correlations in basket CDSs as an example, and the delta-gamma hedge is similar. Assume the number of correlation coefficients is 2, that is to say, we have to hedge the risk caused by ρ_1 and ρ_2. We consider the hedge in the same category. Suppose all the intensities are $\lambda = 0.2$, the recovery rate of the basket CDS is $\delta = 0.1$, and for the underlying single-name CDSs, there are four single-name CDSs whose correlation is $\rho_1 = 0.6$ and one single-name CDS whose correlation is $\rho_2 = 0.5$. In addition, we have some assumptions for the underlying single-name CDSs used as the hedging instruments in our experiments. For Name 1, the recovery rate is $\delta^1 = 0.1$ and correlation is $\rho^1 = \rho_1 = 0.6$, for Name 2, $\delta^2 = 0.1$ and $\rho^2 = \rho_2 = 0.5$, for Name 3, $\delta^3 = 0.15$ and $\rho^3 = \rho_2 = 0.5$ and for Name 4, $\delta^4 = 0.15$ and $\rho^4 = \rho_1 = 0.6$. As mentioned in Model I, let

$$a(t) = V_K(t_J, \boldsymbol{\rho}) - e^{r(t_J - t)} V_K(t, \boldsymbol{\rho}) \tag{62}$$

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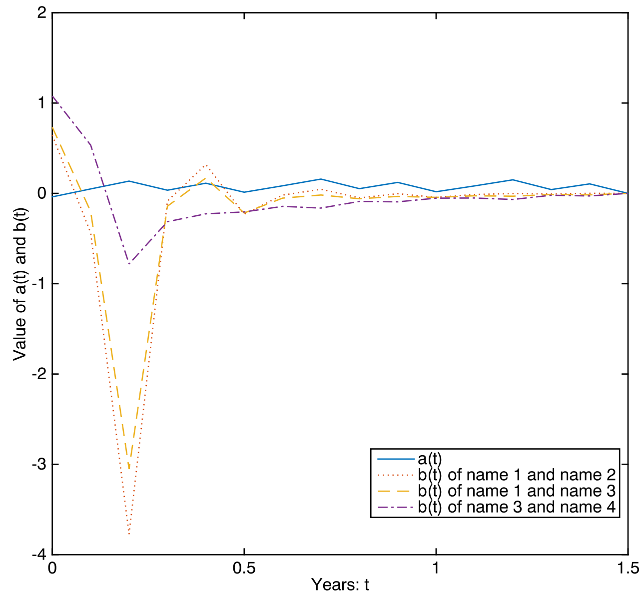

FIG. 3. Hedge using two single CDSs.

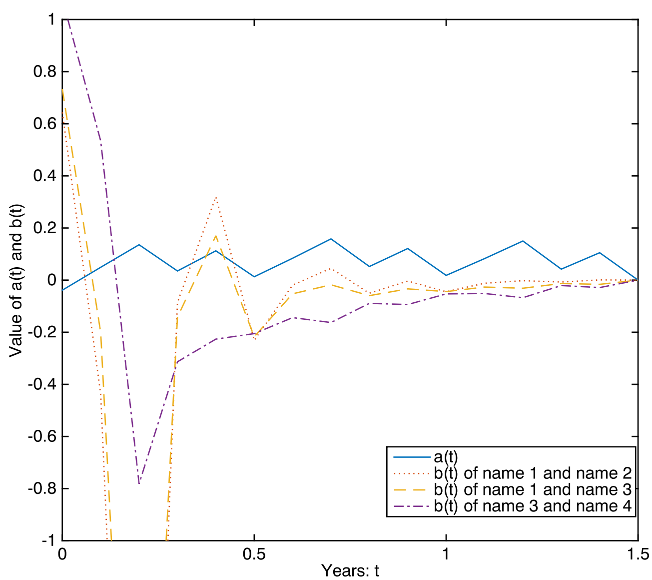

FIG. 4. Hedge using two single CDSs (amplified).

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and

$$b(t) = -\sum_{h=1}^{2}\sum_{j=1}^{J-1} e^{r(t_J-t_j)}\bar{V}_{i_h}(t_j,\boldsymbol{\rho})(\phi^{i_h}(t_{j+1},\boldsymbol{\rho}) - 2\phi^{i_h}(t_j,\boldsymbol{\rho}) + \phi^{i_h}(t_{j-1},\boldsymbol{\rho})) \\ - \sum_{h=1}^{2} e^{r(t_J-t_0)}\bar{V}_{i_h}(t_0,\boldsymbol{\rho})(\phi^{i_h}(t_1,\rho) - 2\phi^{i_h}(t_0,\boldsymbol{\rho})) - \sum_{h=1}^{2}\bar{V}_{i_h}(t_J,\boldsymbol{\rho})\phi^{i_h}(t_{J-1},\boldsymbol{\rho}). \quad (63)$$

Then Fig. 3 depicts $a(t)$ and different $b(t)$ with shifting of the correlation coefficients. The quantitative results are given in Table 3.

By choosing several different single-name CDSs, the hedging experiment becomes more complicated. A zoom-in view of $a(t)$ and $b(t)$ with different combinations of chosen single-name CDSs is given in Fig. 4, which indicates that the hedging effectiveness of the combination of Names 3 and 4 performs the best, and this is also testified by the quantity shown in Table 3. We also observe that as time goes by, the differences between $a(t)$ and $-b(t)$ would be smaller, this phenomenon coincides with intuitive. As time goes by, the underlying survival claims of both single-name CDSs and basket CDSs would tend to be the same with each other.

6. Conclusions

In this paper, we discuss the pricing and hedging of basket CDSs using a Gaussian copula model. The main contribution of the paper is to consider the problem of hedging a basket CDS using their own underlying single-name CDS as hedging instruments. We propose methods that can hedge the price movements in single-name CDSs with respect to the shifting in the correlation. The key advantage of our hedging method is that a smaller number of single-name CDSs may be used as the delta hedging instruments. This may be easier to implement in practice. Three models with different dependent structures are studied and both delta hedge and delta-gamma hedge are examined here. Numerical results are given to show the effectiveness of the hedging methods. Our methods can also be applied to hedge other credit derivatives and be extended to discuss the hedging problems when the default dependence is modeled using other approaches. In such cases, other parameters would be considered, for instance, the CDS spread, which may represent an interesting future research issue.

Funding

This work was supported by Hong Kong Research Grants Council [17301519]; and IMR and RAE Research Fund, Faculty of Science, HKU; and National Natural Science Foundation of China [71601044, 11671158]; and the Fundamental Research Funds for the Central Universities.

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A. Appendix

A.1 *Proof of Proposition 1*

Proof. Based on the given survivorship information, the value of this K^{th}-to-default CDS at time t will be

$$V_K(t,\rho_1,\cdots,\rho_N) = V_K^{def}(t,\rho_1,\cdots,\rho_N) - V_K^{prem}(t,\rho_1,\cdots,\rho_N)$$

$$= \mathbb{E}\left[e^{-r(\tau_K-t)} I_{\{t<\tau_K\le T\}}(1-\delta)\big|\mathcal{H}_t\right] \quad \text{(A1)}$$

$$- \mathbb{E}\left[\sum_{t^M\ge t^{j+1}>\tau_K>t^j>t} I_{\{t^{j+1}>\tau_K>t^j\}}\frac{\tau_K-t^j}{t^{j+1}-t^j} Ye^{-r(\tau_K-t)}\big|\mathcal{H}_t\right],$$

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where I_A is the indicator function, $V_K^{prem}(t,\rho_1,\ldots,\rho_N)$ is the premium payment leg of the K^{th}-to-default swap that denotes the accrued premium payments. The $V_K^{def}(t,\rho_1,\ldots,\rho_N)$ is the price of the default payment leg of the K^{th}-to-default swap.

For arbitrary $s<t$, we denote $(\tau_K > s)$ $G^K(t,s)=\mathbb{P}(\tau_K>t|\mathcal{H}_s)$, then

$$\begin{aligned}
V_K^{prem}(t,\rho_1,\ldots,\rho_N) &= \sum_{t^M\geq t^{j+1}>t^j>t}\frac{Y}{t^{j+1}-t^j}\int_{t^j}^{t^{j+1}}\frac{\partial G^K(x,t)}{\partial x}xe^{-r(x-t)}\mathrm{d}x\\
&\quad-\sum_{t^M\geq t^{j+1}>t^j>t}\frac{Yt^j}{t^{j+1}-t^j}\int_{t^j}^{t^{j+1}}\frac{\partial G^K(x,t)}{\partial x}e^{-r(x-t)}\mathrm{d}x\\
&= \sum_{t^M\geq t^{j+1}>t^j>t}\frac{Y}{t^{j+1}-t^j}\Big[t^{j+1}e^{-r(t^{j+1}-t)}G^K(t^{j+1},t)-t^je^{-r(t^j-t)}G^K(t^j,t)\\
&\quad-\int_{t^j}^{t^{j+1}}G^K(x,t)e^{-r(x-t)}\mathrm{d}x+r\int_{t^j}^{t^{j+1}}G^K(x,t)xe^{-r(x-t)}\mathrm{d}x\Big]\\
&\quad-\sum_{t^M\geq t^{j+1}>t^j>t}\frac{Yt^j}{t^{j+1}-t^j}\Big[G^K(t^{j+1},t)e^{-r(t^{j+1}-t)}-G^K(t^j,t)e^{-r(t^j-t)}\\
&\quad+r\int_{t^j}^{t^{j+1}}G^K(x,t)e^{-r(x-t)}\mathrm{d}x\Big]\\
&= \sum_{t^M\geq t^{j+1}>t^j>t}\Big[YG^K(t^{j+1},t)e^{-r(t^{j+1}-t)}-\frac{Y(1+rt^j)}{t^{j+1}-t^j}\int_{t^j}^{t^{j+1}}G^K(x,t)e^{-r(x-t)}\mathrm{d}x\\
&\quad+\frac{Yr}{t^{j+1}-t^j}\int_{t^j}^{t^{j+1}}G^K(x,t)xe^{-r(x-t)}\mathrm{d}x\Big]
\end{aligned}$$

and

$$\begin{aligned}
V_K^{def}(t,\rho_1,\ldots,\rho_N) &= \mathbb{E}\Big[(1-\delta)I_{(t,T]}(\tau_K)e^{-r(\tau_K-t)}|\mathcal{H}_t\Big]\\
&= (1-\delta)\mathbb{E}\Big[I_{(t,T]}(\tau_K)e^{-r(\tau_K-t)}|\mathcal{H}_t\Big]\\
&= (1-\delta)\int_t^T\frac{\partial(1-G^K(x,t))}{\partial x}e^{-r(x-t)}\mathrm{d}x\\
&= (1-\delta)\int_t^T-\frac{\partial G^K(x,t)}{\partial x}e^{-r(x-t)}\mathrm{d}x\\
&= (1-\delta)\Big[-G^K(x,t)e^{-r(x-t)}|_t^T-r\int_t^TG^K(x,t)e^{-r(x-t)}\mathrm{d}x\Big]\\
&= (1-\delta)\Big[G^K(t,t)-G^K(T,t)e^{-r(T-t)}-r\int_t^TG^K(x,t)e^{-r(x-t)}\mathrm{d}x\Big].
\end{aligned}$$

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Therefore, the value of this K^{th}-to-default CDS at time t is as follows:

$$\begin{aligned} V_K(t,\rho_1,\cdots,\rho_N) &= (1-\delta)\left[G^K(t,t) - G^K(T,t)e^{-r(T-t)} - r\int_t^T G^K(x,t)e^{-r(x-t)}\mathrm{d}x\right] \\ &\quad - \sum_{t^M \geq t^{j+1} > t^j > t}\left[YG^K(t^{j+1},t)e^{-r(t^{j+1}-t)} - \frac{Y(1+rt^j)}{t^{j+1}-t^j}\right. \\ &\quad \left.\int_{t^j}^{t^{j+1}} G^K(x,t)e^{-r(x-t)}\mathrm{d}x + \frac{Yr}{t^{j+1}-t^j}\int_{t^j}^{t^{j+1}} G^K(x,t)xe^{-r(x-t)}\mathrm{d}x\right]. \end{aligned} \tag{A2}$$

□

A.2 *Proof of Proposition 2*

Proof. By Bayes' theorem, we obtain

$$\begin{aligned} G^K(t,s) &= \mathbb{P}(\tau_K > t|\mathcal{H}_s) \\ &= \mathbb{P}(\tau_K > t|\tau_{i_1} = u_1,\ldots,\tau_{i_{k_s}} = u_{k_s},\tau_{j_1} > s,\ldots,\tau_{j_{N-k_s}} > s) \end{aligned}$$

whose numerator can be written as

$$\mathbb{P}(\tau_K > t,\tau_{i_1} \in du_1,\ldots,\tau_{i_{k_s}} \in du_{k_s},\tau_{j_1} > s,\ldots,\tau_{j_{N-k_s}} > s),$$

which is equal to

$$\mathbb{P}(\tau_{i_1} \in du_1,\ldots,\tau_{j_{N-k_s}} > s, \sum_{l=1}^{N-k_s} I_{\{s<\tau_{j_l}\leq t\}} \leq K - k_s - 1)$$

and the denominator is given by

$$\mathbb{P}(\tau_{i_1} \in du_1,\ldots,\tau_{i_{k_s}} \in du_{k_s},\tau_{j_1} > s,\ldots,\tau_{j_{N-k_s}} > s).$$

Then we have

$$\mathbb{P}(\tau^K > t|\mathcal{H}_s) = \frac{\sum_{l=0}^{K-k_s-1}\sum_{I^l}\int_{A_1} f(t_1,\ldots,t_N)|_{D_{k_s}}\mathrm{d}t_{j_1}\ldots\mathrm{d}t_{j_{N-k_s}}}{\int_{B_1} f(t_1,\ldots,t_N)|_{D_{k_s}}\mathrm{d}t_{j_1}\ldots\mathrm{d}t_{j_{N-k_s}}},$$

where

$$\begin{aligned} f(t_1,\ldots,t_N) &= \frac{\partial^N F(t_1,\cdots,t_N)}{\partial t_1\cdots\partial t_N} \\ &= \frac{\partial^N \int\left(\prod_{i=1}^N \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i x}{\sqrt{1-\rho_i^2}}\right)\right)\phi(x)\mathrm{d}x}{\partial t_1\cdots\partial t_N} \\ &= \int\left(\prod_{i=1}^N \frac{\partial}{\partial t_i}\Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i x}{\sqrt{1-\rho_i^2}}\right)\right)\phi(x)\mathrm{d}x \end{aligned}$$

and $f(t_1,\ldots,t_N)|_{D_{k_s}}$ denotes $f(t_{i_1} = u_1,\cdots,t_{i_{k_s}} = u_{k_s},t_{j_1},\cdots,t_{j_{N-k_s}})$. □

A.3 *Proof of Proposition 4*

Proof. According to the definition of delta hedge for a CDS, we need to hedge the shifts in the CDS's value with respect to the shifts in the hedging instrument's value. Therefore, the hedging position in CDS i is equal to

$$dV_K(t,\rho_1,\ldots,\rho_N) = \phi^i(t,\rho_1,\ldots,\rho_N)\cdot d\bar{V}_i(t,\rho_1,\ldots,\rho_N).$$

According to the assumption that $\rho_{n_1},\ldots,\rho_{n_q}$ are distinct and they are selected from $\rho_1,\ldots,\rho_N$, then we have

$$\begin{cases} dV_K(t,\boldsymbol{\rho}) &= \frac{\partial V_K(t,\boldsymbol{\rho})}{\partial\rho_{n_1}}d\rho_{n_1}+\cdots+\frac{\partial V_K(t,\boldsymbol{\rho})}{\partial\rho_{n_q}}d\rho_{n_q} \\ d\bar{V}_i(t,\boldsymbol{\rho}) &= \frac{\partial \bar{V}_i(t,\boldsymbol{\rho})}{\partial\rho_{n_1}}d\rho_{n_1}+\cdots+\frac{\partial \bar{V}_i(t,\boldsymbol{\rho})}{\partial\rho_{n_q}}d\rho_{n_q}. \end{cases}$$

Indeed, $dV_K(t,\boldsymbol{\rho}) = \phi^i(t,\boldsymbol{\rho})\cdot d\bar{V}_i(t,\boldsymbol{\rho})$ means:

$$\left(\frac{\partial V_K(t,\boldsymbol{\rho})}{\partial\rho_{n_1}}d\rho_{n_1},\cdots,\frac{\partial V_K(t,\boldsymbol{\rho})}{\partial\rho_{n_q}}d\rho_{n_q}\right)^T = \phi^i(t,\boldsymbol{\rho})\cdot\left(\frac{\partial \bar{V}_i(t,\boldsymbol{\rho})}{\partial\rho_{n_1}}d\rho_{n_1},\cdots,\frac{\partial \bar{V}_i(t,\boldsymbol{\rho})}{\partial\rho_{n_q}}d\rho_{n_q}\right)^T.$$

We note that only in a special case that this ϕ^i exists. Since $\rho_{n_1},\ldots,\rho_{n_q}$ are distinct, there are q parameters:

$$\frac{\partial V_K(t,\boldsymbol{\rho})}{\partial\rho_{n_j}},\quad j=1,\ldots,q$$

that should be hedged. To make this method applicable to general cases and V_K be well hedged, the number of hedge instruments should also be q. That means we need $\bar{V}_{i_1},\cdots,\bar{V}_{i_q}$ to achieve this goal, then the Delta hedging relationship should be

$$dV_K(t,\boldsymbol{\rho}) = \phi^{i_1}(t,\boldsymbol{\rho})d\bar{V}_{i_1}(t,\boldsymbol{\rho})+\cdots+\phi^{i_q}(t,\boldsymbol{\rho})d\bar{V}_{i_q}(t,\boldsymbol{\rho}). \tag{A3}$$

To be explicit, this can be written as:

$$\begin{pmatrix} \frac{\partial}{\partial\rho_{n_1}}V_K(t,\boldsymbol{\rho}) \\ \vdots \\ \frac{\partial}{\partial\rho_{n_q}}V_K(t,\boldsymbol{\rho}) \end{pmatrix} = H_q(t,\boldsymbol{\rho})\cdot\begin{pmatrix} \phi^{i_1}(t,\boldsymbol{\rho}) \\ \vdots \\ \phi^{i_q}(t,\boldsymbol{\rho}) \end{pmatrix},$$

where $H_q(t,\boldsymbol{\rho})$ is the transpose of the Jacobi matrix. When $H_q(t,\boldsymbol{\rho})$ is invertible, i.e. $\det|H_q(t,\boldsymbol{\rho})|$ is non-zero for all time t we concerned, there exists a unique solution and it can be written in the form:

$$\Phi(t,\boldsymbol{\rho}) = H_q^{-1}(t,\boldsymbol{\rho})\cdot\left(\frac{\partial}{\partial\rho_{n_1}}V_K(t,\boldsymbol{\rho}),\cdots,\frac{\partial}{\partial\rho_{n_q}}V_K(t,\boldsymbol{\rho})\right)^T.$$

The element $\phi^{i_j}(t,\boldsymbol{\rho})$ obtained is the corresponding hedging position for the single-name-i_j CDS. □

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A.4 *Proof of Proposition 5*

Proof. Here, the conditions for both delta hedge and gamma hedge should satisfy. The first-order total differentiation has been given before, the second-order total differentiation is in the following form:

$$d^2V_K = d(dV_K) = \frac{\partial^2 V_K}{\partial \rho_{n_1}^2} d\rho_{n_1}^2 + \cdots + \frac{\partial^2 V_K}{\partial \rho_{n_q}^2} d\rho_{n_q}^2 + 2\sum_{i \neq j} \frac{\partial^2 V_K}{\partial \rho_{n_i} \partial \rho_{n_j}} d\rho_{n_i} d\rho_{n_j}, \tag{A4}$$

where $i, j = 1, \ldots, q$, and actually, the number of this term is $q + \frac{q(q-1)}{2}$, the number of term dV_K is q. Since the conditions for both delta hedge and gamma hedge should satisfy, the total number of parameters that we need to hedge is then $s = 2q + \frac{q(q-1)}{2}$. Thus, we need s single-name CDSs as hedge instruments, and we denote them as single-name-i_1 CDS, $\cdots$ single-name-i_s CDS. Therefore, the two conditions to be satisfied are:

$$\begin{cases} dV_K(t, \boldsymbol{\rho}) = \bar{\phi}_{i_1}(t, \boldsymbol{\rho}) d\bar{V}_{i_1}(t, \boldsymbol{\rho}) + \cdots + \bar{\phi}_{i_s}(t, \boldsymbol{\rho}) d\bar{V}_{i_s}(t, \boldsymbol{\rho}) \\ d^2V_K(t, \boldsymbol{\rho}) = \bar{\phi}_{i_1}(t, \boldsymbol{\rho}) d^2\bar{V}_{i_1}(t, \boldsymbol{\rho}) + \cdots + \bar{\phi}_{i_s}(t, \boldsymbol{\rho}) d^2\bar{V}_{i_s}(t, \boldsymbol{\rho}). \end{cases} \tag{A5}$$

Similarly, the above two conditions can be written in this way:

$$\begin{pmatrix} \frac{\partial}{\partial \rho_{n_1}} V_K(t, \boldsymbol{\rho}) \\ \vdots \\ \frac{\partial}{\partial \rho_{n_q}} V_K(t, \boldsymbol{\rho}) \\ \frac{\partial^2}{\partial \rho_{n_1}^2} V_K(t, \boldsymbol{\rho}) \\ \vdots \\ \frac{\partial^2}{\partial \rho_{n_q}^2} V_K(t, \boldsymbol{\rho}) \\ 2\frac{\partial^2}{\partial \rho_{n_1} \partial \rho_{n_2}} V_K(t, \boldsymbol{\rho}) \\ \vdots \\ 2\frac{\partial^2}{\partial \rho_{n_i} \partial \rho_{n_j}} V_K(t, \boldsymbol{\rho}) \\ \vdots \\ 2\frac{\partial^2}{\partial \rho_{n_{q-1}} \partial \rho_{n_q}} V_K(t, \boldsymbol{\rho}) \end{pmatrix} = \bar{H}_s(t, \boldsymbol{\rho}) \cdot \begin{pmatrix} \bar{\phi}_{i_1}(t, \boldsymbol{\rho}) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \bar{\phi}_{i_s}(t, \boldsymbol{\rho}) \end{pmatrix}.$$

Only when $\bar{H}_s(t, \boldsymbol{\rho})$ is invertible, that is to say $\det |\bar{H}_s(t, \boldsymbol{\rho})|$ is non-zero holds for all t we concerned, there exists a unique solution of

$$\bar{\Phi}(t, \boldsymbol{\rho}) = \bar{H}_s^{-1}(t, \boldsymbol{\rho}) \cdot \left(\frac{\partial V_K}{\partial \rho_{n_1}}, \cdots, \frac{\partial V_K}{\partial \rho_{n_q}}, \frac{\partial^2 V_K}{\partial \rho_{n_1}^2}, \cdots, \frac{\partial^2 V_K}{\partial \rho_{n_q}^2}, 2\frac{\partial^2 V_K}{\partial \rho_{n_1} \partial \rho_{n_2}} \cdots, 2\frac{\partial^2 V_K}{\partial \rho_{n_i} \partial \rho_{n_j}}, \cdots 2\frac{\partial^2 V_K}{\partial \rho_{n_{q-1}} \partial \rho_{n_q}} \right)^T$$

satisfying the conditions for the delta-gamma hedge. Similarly, the element $\bar{\phi}^{i_h}(t, \boldsymbol{\rho})$ obtained from this formula is the corresponding hedging position for the single-name-i_h CDS ($h = 1, 2, \ldots, s$). □

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