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Modeling Credit Risk with Hidden Markov Default Intensity

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Abstract This paper investigates the modeling of credit default under an interactive reduced-form intensity-based model based on the Hidden Markov setting proposed in Yu et al. (Quant Finance 7(5):781–794, 2017). The intensities of defaults are determined by the hidden economic states which are governed by a Markov chain, as well as the past defaults. We estimate the parameters in the default intensity by using Expectation–Maximization algorithm with real market data under three different practical default models. Applications to pricing of credit default swap (CDS) is also discussed. Numerical experiments are conducted to compare the results under our models with real recession periods in US. The results demonstrate that our model is able to capture the hidden features and simulate credit default risks which are critical

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in risk management and the extracted hidden economic states are consistent with the real market data. In addition, we take pricing CDS as an example to illustrate the sensitivity analysis.

Keywords Credit default swap (CDS) · Credit risk · Expectation–maximization (EM) algorithm · Intensity models

1 Introduction

Modeling credit risk plays an important role in credit risk management. Much attention has been given to it especially after the global financial crisis. Popular models adopted in the finance industry can be divided into two major categories: structural firm value models and reduced-form intensity-based models. The structural firm value model was pioneered by Black and Scholes (1973) and Merton (1974). The key idea is to model the default of a firm by using its asset value, and adopt a geometric Brownian motion to describe the asset value. When the asset value falls below a certain prescribed level, the default of the firm is deemed to be triggered. The reduced-form intensity-based model was pioneered by Jarrow and Turnbull (1995) and Madan and Unal (1998). The main idea is to consider the defaults as exogenous processes and describe their occurrences with Poisson processes and their variants. A middle ground model between the structural models and the reduced-form models was introduced by Cathcart and El-Jahel (1998) and it is called the signaling approach. They assumed that the default occurs when some signaling process which captures the factors that affect its default probability rather than the value of the assets of the firm hits a lower constant default barrier. Besides, some hybrid credit risk models were also proposed. For example, Cathcart and El-Jahel (2004) further developed an approach which combines the structural value models and the reduced-form models by allowing expected and unexpected defaults, and analytical solutions for defaultable bonds under this model was derived. The default occurrences in this model depend on a signaling variable which is interpreted as the credit quality of the issuer, i.e., its rating, hits the threshold or a hazard rate linearly dependent on the default-free interest rate. Ballestra and Pacelli (2014) introduced a new hybrid model in which the default intensity is described by a stochastic differential equation coupled with the process of the obligor's asset value, and derived an closed-form approximation solution.

In this paper, we investigate the credit risk model and calculation methods proposed in Yu et al. (2017). We consider adopting reduced-form intensity-based model which has been widely adopted to solve portfolio default risk problems. A variety of intensity models have been developed using ratings and corresponding default intensities for the evaluation of credit risk since the 1990s. The form of the default intensities includes constant intensity in Jarrow and Turnbull (1995), intensity depends on the stock price in Madan and Unal (1998), and intensity depends on some state variable in Lando (1998), etc. The rating process is typically done by the rating agencies and its migration processes could be modeled by the semi-Markov processes. For more details, interested readers may refer to Trueck and Rachev (2009) and D'Amico et al. (2017). In terms of the way of describing the dependent defaults, to be more specific, reduced-form intensity-based models can be further divided into top-down models and bottom-up models. Top-down models consider modeling the default times at the portfolio level without reference to the intensities of individual entities. To recover the individual entity's intensity, one can employ random thinning and other methods. Some works related to top-down methods include Brigo et al. (2006), Cont and Minca (2011), Davis and Lo (2001), Giesecke et al. (2011) and Longstaff and Rajan (2008), etc. Bottom-up models, on the other hand, focus on modeling the default intensities of individual reference entities and their aggregation to form a portfolio default intensity. Some works related include Duffie and Garleanu (2001), Duffie et al. (2006), Giesecke and Goldberg (2004), Gu et al. (2013), Jarrow and Yu (2001), Schönbucher and Schubert (2001), Yu (2007) and Yu et al. (2017), etc. This paper uses a bottom-up model.

Madan and Unal (1998) decomposed the default debt into survival security which faces timing risk and default security which faces recovery risk, and derived explicit prices and estimation methods for these two risks. Based on the credit risk model proposed in Madan and Unal (1998), Ballestra et al. (2017) obtained an accurate and fast quasi-analytical approximation formula for the survival probability by using a Laplace transform approach, and showed this formula could also be used for pricing CDSs. They also pointed out that the model by Madan and Unal (1998) is rather consistent with the real market data. Yu (2007) extended the model proposed by Lando (1998) and applied it to multiple defaults and their correlation. The distribution of default times with interacting intensities can then be simulated with the total hazard construction method proposed by Norros (1986) and Shaked and Shanthikumar (1987). Based on this method, Zheng and Jiang (2009) then derived closed-form formulas for the multiple default distributions under their contagion model. Gu et al. (2013) proposed an ordered default rate method to calculate the distribution of ordered default times with recursive formula. To extend, Gu et al. (2016) also applied reduced-form model to study the probability distributions of the economic and recored default times. Under a specific form of default intensities, Gu et al. (2014) further proposed a hidden Markov reduced-form default model and extracted the hidden process with observable information. Based on this hidden model, Yu et al. (2017) generalized it and developed a reduced-form intensity-based hidden credit model which are widely applicable to various type of default intensities. Since filtering is a key step for Hidden Markov Model (HMM), several different filtering methods could be found in the literature. A popular and prominent method is based on reference probability approach which starts with a reference probability measure under which the observed dynamics become simpler and do not depend on the hidden state processes. For example, Elliott et al. (2008), (2014), Frey and Runggaldier (2010, 2011), Frey and Schmidt (2011) and Elliott and Siu (2013), etc.

In Yu et al. (2017), a flexible method for extracting the hidden process without constraints on the dynamic of the model is derived with the similar idea from the filtering method in Gu et al. (2014) by using moment generating function. With the extracted hidden states, they employed the total construction method proposed in Yu (2007) and developed closed-form formulas to calculate the joint default distribution. When the intensities are homogeneous and symmetric, with the analytical formula of joint default distribution, analytical algorithms are also derived for the calculation of ordered default distributions.

In this paper, with the methods for default distributions introduced in Yu et al. (2017), we calibrate the reduced-form intensity-based hidden credit model and conduct empirical study using real market data. Numerical experiments in Yu et al. (2017) only considered two types of default intensities as examples to illustrate the methods. Here we discuss three more complicated default intensities models which include a no decay case, a no-impact-decay case and impact-decay case. The default intensity models we proposed are more realistic when compare to those in Yu et al. (2017). Furthermore, estimation method, which is not discussed in Yu et al. (2017), is proposed here. The parameters in our hidden credit risk models are estimated by using the Expectation–Maximization (EM) algorithm with real market data. Hypothesis testings are also conducted to support the estimation results.

The rest of the paper is structured as follows. Section 2 gives a snapshot of the interacting intensity-based default model with hidden Markov process. Section 3 explicitly presents the three default intensities models and the procedure of estimating parameters with EM Algorithms. In Sect. 4, one application of default risk model: pricing Credit Default Swap (CDS) is discussed and formulas for sensitivity analysis are also derived. Besides, experiments to demonstrate the sensitivity analysis are also given. Section 5 provides the numerical experiments with real-world datasets to illustrate the proposed methods discussed. Section 6 concludes the paper.

2 Model Setup

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space where P is a risk-neutral probability measure, which is assumed to exist. Suppose there are K interacting entities, and we let $N_i(t) := 1_{\{\tau_i \leq t\}}$, where τ_i is a stopping time, representing the default time of credit name i, for $i = 1, 2, \ldots, K$ and 1_A is the indicator function. The indicator function gives the value "1" if the statement A is true, and "0", otherwise. Suppose we have an underlying state process $(X_t)_{t\geq 0}$ describing the dynamics of the economic condition. Let $\mathcal{F}_t^X := \sigma(X_s, 0 \leq s \leq t) \vee \mathcal{N}$ where $\mathcal{N}$ represents all the null subsets of Ω in $\mathcal{F}$ and $\mathcal{C}_1 \vee \mathcal{C}_2$ is the minimal σ-algebra containing both the σ-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$. We also let $\mathcal{H}_t := \mathcal{F}_t^X \vee \mathcal{F}_t^N$ where

$$\mathcal{F}_t^N = \mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \ldots \vee \mathcal{F}_t^K \quad \text{and} \quad \mathcal{F}_t^i := \sigma\left(1_{\{\tau_i \leq s\}}, 0 \leq s \leq t\right) \vee \mathcal{N}.$$

We assume that for each $i = 1, 2, \ldots, K$, $N_i(t)$ possesses a nonnegative, $\{\mathcal{H}_t\}_{t\geq 0}$-adapted, intensity process λ_i satisfying

$$E\left(\int_0^t \lambda_i(s)ds\right) < \infty, \quad t \geq 0, \tag{1}$$

such that the compensated process

$$M_i(t) := N_i(t) - \int_0^{t\wedge\tau_i} \lambda_i(s)ds \ , \quad t \geq 0, \tag{2}$$

is an $(\{\mathcal{H}_t\}_{t\geq 0}, P)$-martingale. Note that after the default time τ_i, $N_i(t)$ will stay at the value one, so there is no need to compensate for $N_i(t)$ after time τ_i.

For all the market participants, we assume that they cannot observe the underlying process $(X_t)_{t\geq 0}$ directly. Instead, they observe the process $(Y_t)_{t\geq 0}$, revealing the delayed and noisy information of $(X_t)_{t\geq 0}$, and also observe the default process $(N_t^i)_{t\geq 0}$. For example, $Y(t)_{t\geq 0}$ could be linked to the interest rate, the situation of defaults occurred in the market, etc, which could reveal the economics situations. In our numerical experiments, we assume if the number of defaults exceed a prescribed value, it gives people the information that the current economic situation is bad and therefore we set a value for the observable process $Y(t)_{t\geq 0}$ which represents the information. Hence, the common information set available to the market participants at time t is

$$\mathcal{F}_t := \mathcal{F}_t^Y \vee \mathcal{F}_t^N \quad \text{where} \quad \mathcal{F}_t^Y := \sigma(Y_s, 0 \leq s \leq t) \vee \mathcal{N}.$$

We further assume that $(X_t)_{t\geq 0}$ is an "exogenous" process to $(N_t^i)_{t\geq 0}$, for $i = 1, 2, \ldots, K$. This means for any t, the σ-fields $\mathcal{F}_\infty^X$ and $\mathcal{F}_t^N$ are conditionally independent given $\mathcal{F}_t^X$ and $P(\tau_i \neq \tau_j) = 1, i \neq j$.

To simplify our discussion, throughout the paper, we suppose that $(X_t)_{t\geq 0}$ is a two-state Markov chain taking a value in $\{x_0, x_1\}$. We assume that the transition rates of the chain for "$x_0 \to x_1$" and "$x_1 \to x_0$" are θ_0 and θ_1, respectively. The observable process $(Y_t)_{t\geq 0}$ is again a two-state Markov chain taking a value in $\{y_0, y_1\}$, with transition rates depending on X_t, i.e., $\eta_0(X_t)(y_0 \to y_1)$ and $\eta_1(X_t)(y_1 \to y_0)$, where η_0 and η_1 are real-valued functions. At time 0, we suppose that X_0 is in state x_0 and Y_0 is in state y_1. Since the methods introduce in Yu et al. (2017) can still be applicable when the Markov chains X and Y have more than two states, i.e., finite many states though more complicated notations may involve. We remark that the discussion in this paper could also be applicable to more states.

3 Default Distributions

To facilitate our discussion and specify the form of the intensities, we give the following notations. Suppose that at time t, N_t^D defaults have already occurred at $t_1, t_2, \ldots, t_{N_t^D}$ such that

$$0 = t_0 < t_1 < \cdots < t_{N_t^D} \leq t.$$

Then we denote $T_{N_t^D} = (t_1, \ldots, t_{N_t^D})$ the ordered N_t^D default times and $I_{N_t^D} = (j_1, \ldots, j_{N_t^D})$ the corresponding N_t^D defaulters. Each process λ_i $(i = 1, \ldots, K)$, is $\{\mathcal{H}_t\}_{t\geq 0}$-predictable, that is to say $\lambda_i(t)$ is known given information about the chain X and all the default processes prior to time t. Then the intensity of τ^i can be written as $\lambda_t^i = \lambda_i(t|I_{N_t^D}, T_{N_t^D}, X_t)$ where X_t is the state of the chain X at time t. Note that $(I_{N_t^D}, T_{N_t^D}, X_t) \in \mathcal{H}_t$.

Here we consider three different default intensity models:

(i) Decay Model I:

$$\lambda_i(t) = \exp\left(a + \left(c\sum_{j\neq i} 1_{\{\tau_j \le t\}}\right)\cdot e^{-t/\gamma} + b\cdot X(t)\right) \tag{3}$$

(ii) Decay Model II:

$$\lambda_i(t) = \exp\left(a + c\sum_{j\neq i} 1_{\{\tau_j \le t\}}\cdot e^{-(t-\tau_j)/\gamma} + b\cdot X(t)\right) \tag{4}$$

(iii) No-decay Model:

$$\lambda_i(t) = \exp\left(a + c\sum_{j\neq i} 1_{\{\tau_j \le t\}} + b\cdot X(t)\right). \tag{5}$$

Here a, b and c are unknown parameters which could be estimated with real default data, γ is a parameter to adjust the default scale. All of the three models established in this paper are in the form of exponential function which are different from the models discussed in Yu et al. (2017). With this exponential form, the default intensities in the Poisson process could be guaranteed to be positive regardless of the values of parameters in the models.

For the estimation of the parameters a, b and c in the intensity models, we employ the EM Algorithm. To demonstrate the idea, we assume that the transition rates of jump process Y is

$$\eta_0(x) = \begin{cases} p_{01}, & x = x_0 \\ p_{11}, & x = x_1 \end{cases} \quad \text{and} \quad \eta_1(x) = \begin{cases} p_{00}, & x = x_0 \\ p_{10}, & x = x_1. \end{cases} \tag{6}$$

It is possible to define the process in continuous time. However, to simplify the notation for our filtering, smoothing, and the EM algorithm, we assume that the process is defined on a discrete set of evenly spaced lattice points. We divide the interval $[0, T]$ into L intervals of equal width $\delta = T/L$, then $t = l\cdot\delta$ for $l = 0, 1, \ldots, L$. Based on EM algorithm, we could derive the corresponding likelihood function and estimate the related parameters. We take decay Model I as an example. Since γ is determined by the datasets, we simply assume that it is 1 for convenience. The likelihood function could be written as follows:

$$
\begin{aligned}
p(N, J, H|X) = \prod_t & \left[\exp\left(\left(a + c \cdot \left(\sum_{i \in N^+} 1_{\{\tau_i \le t\}}\right)\right) \cdot e^{-t} + b \cdot (X(t))\right)\right]^{N(t)} \\
& \cdot \frac{1}{(N(t)!)} \exp\left[-\exp\left(\left(a + c \cdot \left(\sum_{i \in N^+} 1_{\{\tau_i \le t\}}\right)\right) \cdot e^{-t} + b \cdot (X(t))\right)\right] \\
& \cdot (p_{00} + (p_{10} - p_{00}) \cdot X(t))^{J(t) \cdot (H(t))} \\
& \cdot (p_{01} + (p_{11} - p_{01}) \cdot X(t))^{J(t) \cdot (H(t)+1)} \\
& \cdot (1 - p_{01} + (p_{01} - p_{11}) \cdot X(t)^{(1-J(t)) \cdot (H(t)+1)} \\
& \cdot (1 - p_{00} + (p_{00} - p_{10}) \cdot X(t)^{(1-J(t)) \cdot (H(t))}
\end{aligned} \tag{7}
$$

where $N(t)$ is the default processes. Here $J(t)$ indicates whether a jump occurred at t, $H(t)$ represents the cumulative numbers of jumps at t. With the assumed transition rates for jump process, we can then derive the likelihood calculation for jump process, i.e.,

$$
\begin{aligned}
j(J, H|X) = \; & (p_{00} + (p_{10} - p_{00}) \cdot X(t))^{J(t) \cdot (H(t))} \\
& \cdot (p_{01} + (p_{11} - p_{01}) \cdot X(t))^{J(t) \cdot (H(t)+1)} \\
& \cdot (1 - p_{01} + (p_{01} - p_{11}) \cdot X(t)^{(1-J(t)) \cdot (H(t)+1)} \\
& \cdot (1 - p_{00} + (p_{00} - p_{10}) \cdot X(t)^{(1-J(t)) \cdot (H(t))}.
\end{aligned} \tag{8}
$$

If $X(t) = 0$, then $p_{00} + (p_{10} - p_{00}) \cdot X(t) = p_{00}$, if $X(t) = 1$, then $p_{00} + (p_{10} - p_{00}) \cdot X(t) = p_{10}$, etc, which embody the transition rates for jump process. While the rest parts of $p(N, J, H|X)$ is the likelihood function of the default process. Of course, with different transition rates of jump process Y, we can derive the corresponding likelihood functions. Taking logarithm at the right-hand side of the above function, the log-likelihood function related to the parameters a, b and c is given by

$$
\begin{aligned}
l(N|X) = \; & a\big(N(\tau_1)\exp\{-\tau_1\} + \cdots + N(\tau_n)\exp\{-\tau_n\}\big) \\
& + c\big(D_1 N(\tau_2)\exp\{-\tau_2\} + D_{n-1} N(\tau_n)\exp\{-\tau_n\}\big) \\
& + b\big(N(\tau_1)X_{\tau_1} + \cdots + N(\tau_n)X_{\tau_n}\big) - \textstyle\sum_0^{\tau_1} \exp(a\exp\{-t\} + b(X_t)\big) \\
& - \cdots - \textstyle\sum_{\tau_n}^{T} \exp\big(a\exp\{-t\} + cD_n\exp\{-t\} + b(X_t)\big) - \log(N(t)!)\big)
\end{aligned} \tag{9}
$$

where $D_1, \ldots, D_i, \ldots, D_n$ represent the cumulative default times by the ith default time, that is to say $D_i = \sum_{j \in N^+} 1_{\{\tau_j \le \tau_i\}}$. According to the EM algorithm, we compute the E-step first, which is the expectation of the log-likelihood function given the parameter estimated from the last iteration. Then in the M-step, we take their derivatives with respect to a, b and c, and let the individual derivatives equal to zero. Therefore, we obtain the following equations.

(i) For parameter a:

$$
\begin{aligned}
& N(\tau_1)\exp(-\tau_1) + \cdots + N(\tau_n)\exp(-\tau_n) \\
& = E\Big[\textstyle\sum_t^{\tau_1} \exp(-t)\exp(a\exp\{-t\} + b(X_t)) + \cdots \\
& \quad + \textstyle\sum_{\tau_n}^{T} \exp(-t)\exp(a\exp\{-t\} + cD_n\exp\{-t\} + b(X_t))\Big].
\end{aligned} \tag{10}
$$

(ii) For parameter b:

$$\begin{aligned} &N(\tau_1)X_{\tau_1} + \cdots + N(\tau_n)X_{\tau_n} \\ &= E\Big[\textstyle\sum_{t=0}^{\tau_1} \exp(a\exp\{-t\} + b(X(t)))X(t) + \cdots \\ &\quad + \textstyle\sum_{t=\tau_n}^{T} \exp((a + cD_n)\exp\{-t\} + b(X(t)))X(t)\Big]. \end{aligned} \tag{11}$$

(iii) For parameter c:

$$\begin{aligned} &N(\tau_2)D_1\exp(-\tau_2) + \cdots + N(\tau_n)D_{n-1}\exp(-\tau_n) \\ &= E\Big[\textstyle\sum_{\tau_1}^{\tau_2} D_1\exp(-t)\exp(a\exp\{-t\} + cD_1\exp\{-t\} + b(X_t)) + \cdots \\ &\quad + \textstyle\sum_{\tau_n}^{T} D_n\exp(-t)\exp(a\exp\{-t\} + cD_n\exp\{-t\} + b(X_t))\Big]. \end{aligned} \tag{12}$$

Since the path of X is unobservable, while the path of Y and N_i, $(i = 1, \ldots, K)$ are observable, we can exploit the relationship between X, Y and N_i, $i = 1, \ldots, K$. To find the probability law of X, we apply the recursive method proposed in Yu et al. (2017) to calculate the conditional probability $P(X_t = x_i|\mathcal{F}_t)$, $(i = 0, 1, t \geq 0)$.

4 Sensitivity Analysis

In this section, we consider pricing Credit Default Swaps (CDS) and conduct sensitivity analysis to study the effect of the model parameters. Assume that the buyer of the CDS agrees to pay premiums to the seller continuously over time at a fixed rate until the expiration time of the CDS contract. For instance, we consider the following intensity model (Decay Model I):

$$\lambda_i(t) = \exp\left(a + \left(c\sum_{j\neq i} 1_{\{\tau_j \leq t\}}\right)\cdot e^{-t/\gamma} + b\cdot X(t)\right). \tag{13}$$

We consider the sensitivity analysis of the following two cases of CDS. To be more specific, we consider a first-to-default basket CDS contact and a second-to-default CDS contact individually. For the first-to-default CDS, if any one entity out of the portfolio of first-to-default basket CDS contact default prior to the expiry time, then \$1 will be paid. Similar to the second-to-default CDS contact, if any two entities default prior to the expiry time, then \$1 will be paid. For simplicity, this payment only occurs at the expiry time, but the payment of premium occurs at the initial time.

Let y be the fixed premium rate, and suppose the issue time of the swap contract is 0, the expiry time is T, and we are at time s, then the present value of the premium payment from the buyer should be

$$E\left(\int_0^T e^{(-rs)} y 1_{\{\tau_1 > s\}} ds\right) \tag{14}$$

where τ_1 represents the first default time and r denotes the interest rate. This means if any one of the entities in the portfolio defaults, the buyer of the CDS contract would stop paying the premium.

1. First-to-default CDS: the present the value of the seller should be $E\left(e^{(-rT)}1_{\{\tau_1 \le T\}}\right)$. Therefore, the premium rate is

$$y_1 := y = \frac{E\left(e^{(-rT)}1_{\{\tau_1 \le T\}}\right)}{E\left(\int_0^T e^{(-rs)}1_{\{\tau_1 > s\}}ds\right)} \tag{15}$$

   and we denote this y as y_1 to distinguish the one in the second case.
2. Second-to-default CDS: the present the value of the seller should be $E\left(e^{(-rT)}1_{\{\tau_2 \le T\}}\right)$. Therefore, the premium rate is

$$y_2 := y = \frac{E\left(e^{(-rT)}1_{\{\tau_2 \le T\}}\right)}{E\left(\int_0^T e^{(-rs)}1_{\{\tau_1 > s\}}ds\right)} \tag{16}$$

   for this second-to-default CDS case.

Assume there is no jump observed in the observable process $Y(t)$ before the expiry time T. Without loss of generality, we further assume that process $X(t)$ does not jump neither before T, which means this stochastic process is degenerate and $X(t) = k, 0 \le t \le T$ for some constant k, to simplify our discussion. Therefore, we have the following formulas

$$y_1 = \frac{e^{(-rT)}\left(r+\lambda_1\right)\left(1-e^{-\lambda_1 T}\right)}{1-e^{(-(r+\lambda_1)T)}} \quad \text{and} \quad y_2 = \frac{\beta \cdot e^{(-rT)}\left(r+\lambda_1\right)}{1-e^{(-(r+\lambda_1)T)}} \tag{17}$$

where

$$\begin{aligned} \beta &= P\left(\tau_2 \le T\right) = 1 - P\left(\tau_2 > T\right) \\ &= P\left(T < \tau_1\right) + P\left(\tau_1 \le T < \tau_2\right) \\ &= e^{-\lambda_1 T} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 T}\left(e^{(\lambda_2-\lambda_1)T} - 1\right) \end{aligned} \tag{18}$$

and $\lambda_1 = \exp\left(a + b \cdot X(t)\right)$, $\lambda_2 = \exp\left(a + c \cdot e^{-t/\gamma} + b \cdot X(t)\right)$.

Given the assumption that the interest rate is $r = 3\%$, expiry time $T = 1$ year, $\gamma = 60$, the process $X(t) = 0.5$ before the expiry time, and the initial parameters are $a = -1.5$, $b = 2.5$ and $c = 0.02$, we conducted the following sensitivity analysis with respect to parameters a, b and c under two CDS cases, see, Figs. 1 and 2.

As the parameters increases, on the whole, the premium rates would increase accordingly. In other words, when the weight of influence factors become bigger, the premium rates would become higher. In the first-to-default basket CDS case, premium y does not change as parameter c changes. This is because in this case, the default intensity does not depend on c.

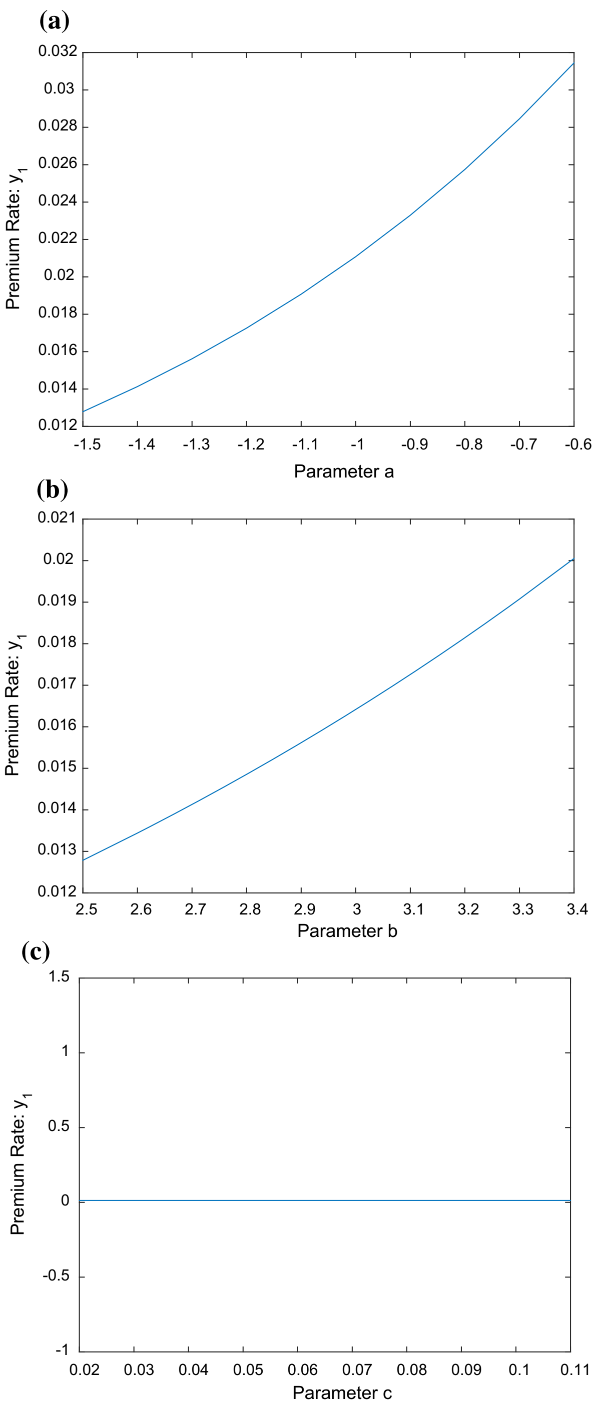

Fig. 1 Change of premium y_1 with coefficients under 1st-to-default CDS. **a** premium y_1 change with respect to a, **b** premium y_1 change with respect to b, **c** premium y_1 change with respect to c

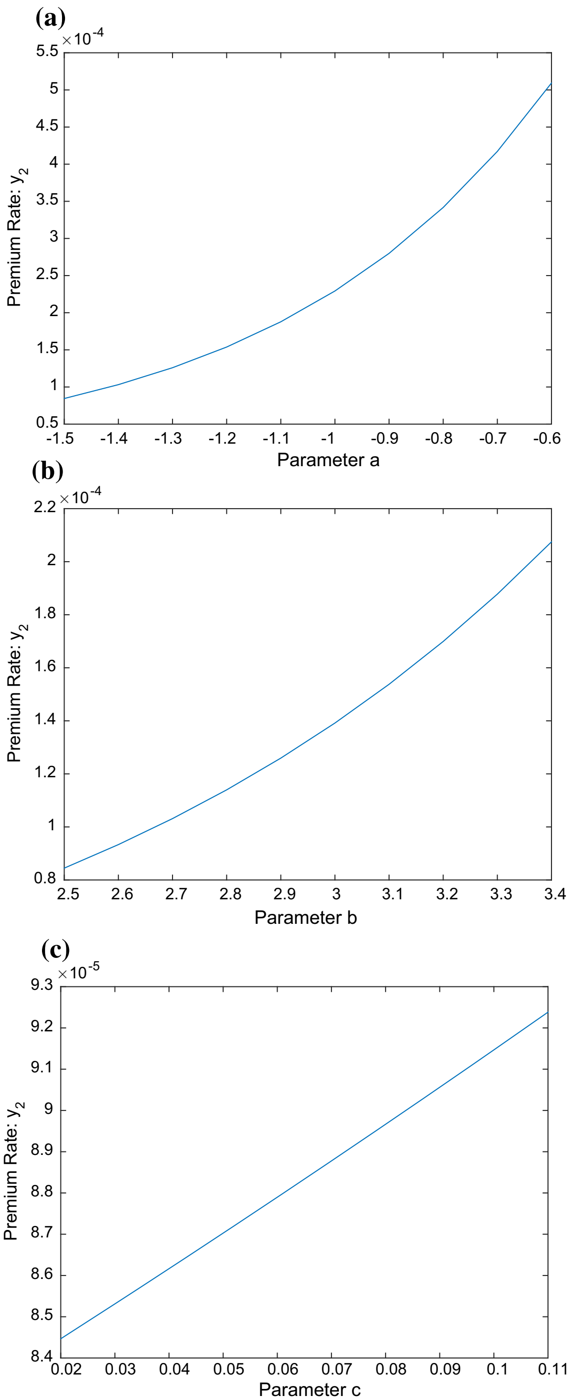

Fig. 2 Change of premium y_2 with coefficients under 2nd-to-default CDS. **a** premium y_2 change with respect to a, **b** premium y_2 change with respect to b, **c** premium y_2 change with respect to c

5 Default Data and Numerical Experiments

In this section, we conduct some numerical experiment with the real market data. In our experiments, we assume $(x_0, x_1) = (0, 1)$ and $(y_0, y_1) = (0, 1)$ and we let the transition rates be $\theta_0 = 0.1$ and $\theta_1 = 0.1$, the initial state $x_0 = 0$. For the observable chain Y_t, we set the transition rates as

$$\eta_0(x) = \begin{cases} 0.3, & x = x_0 \\ 0.1, & x = x_1 \end{cases} \quad \text{and} \quad \eta_1(x) = \begin{cases} 0.1, & x = x_0 \\ 0.3, & x = x_1. \end{cases} \tag{19}$$

The initial state is $y_1 = 1$ as we assumed. For the parameters estimation under EM algorithm, KolmogorovSmirnov (K–S) test is employed to measure the agreement between the model and real data sequences. We consider applying the p value to test the results, and the level of significance is assumed to be 5%. For all the three intensity models, we assume $\gamma = 70$.

In our numerical experiments, we adopt four real defaults sequences observed in the period 1981–2002 used in Giampieri et al. (2005). They are taken from four industrial sectors: consumer/service, energy and natural resources, leisure time/media and transportation. The number of default events could be seen in Fig. 3 for the four different sectors. Since the real market default data are seasonal data in the period of 1981–2002, there are 88 data points in each default sequence as the time length is $T = 22$. The number L of time intervals in the EM algorithm is chosen to be 88 to use all of the data available and to make sure the step size δ is seasonal which is 1/4. We take the first three sequences, i.e., (i) Consumer/Service, (ii) Energy and Natural Resources, and (iii) Leisure Time/Media to form a combination. If there are two or more sequences defaulted among these three sequences, then $Y = 1$ which represents the "bad" economic state, and $Y = 0$ represents the "good" state.

Under Decay Model I:

$$\lambda_i(t) = \exp\left(a + \left(c \sum_{j \neq i} 1_{\{\tau_j \leq t\}}\right) \cdot e^{-t/\gamma} + b \cdot X(t)\right) \tag{20}$$

the estimated parameters are $\bar{a} = -2.76$, $\bar{b} = 2.00$, $\bar{c} = 0.155$ and the p value is 0.0690. Figure 4 shows the results.

Under Decay Model II:

$$\lambda_i(t) = \exp\left(a + c \sum_{j \neq i} 1_{\{\tau_j \leq t\}} \cdot e^{-(t-\tau_j)/\gamma} + b \cdot X(t)\right) \tag{21}$$

the estimated parameters are $\bar{a} = -5.14$, $\bar{b} = 4.86$, $\bar{c} = 0.032$ and the p value is 0.0700. Figure 5 shows the results.

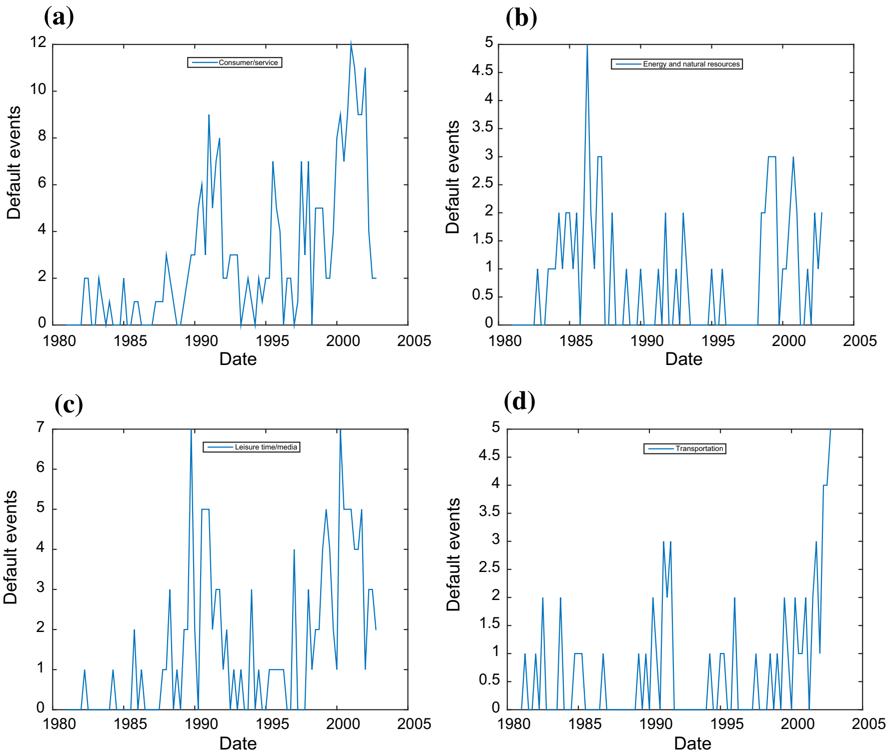

Fig. 3 The default sequences of four sectors. **a** The default sequences of consumer/service. **b** The default sequences of energy and natural resources. **c** The default sequences of leisure time/media. **d** The default sequences of transportation

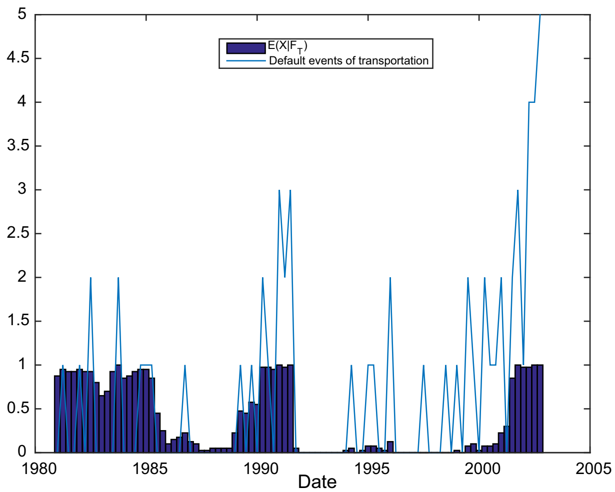

Fig. 4 $E(X|F_T)$ under the Decay Model I

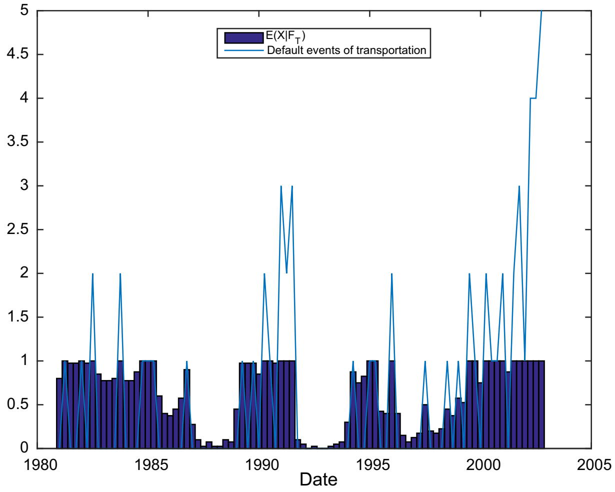

Fig. 5 $E(X|F_T)$ under the Decay Model II

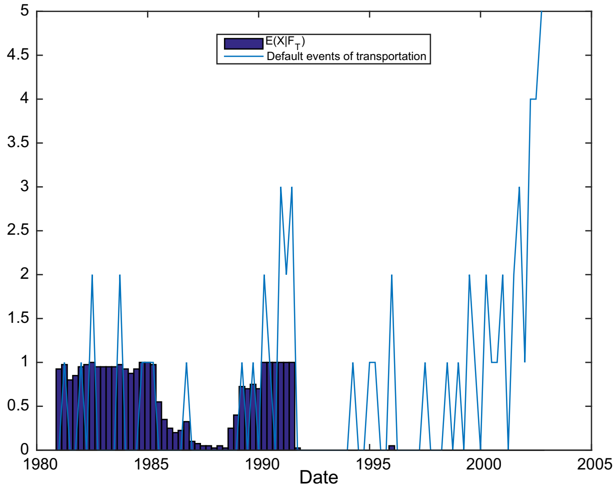

Fig. 6 $E(X|F_T)$ under the No-decay Model

Under No-decay Model:

$$\lambda_i(t) = \exp\left(a + c\sum_{j\neq i} 1_{\{\tau_j \leq t\}} + b \cdot X(t)\right) \tag{22}$$

the estimated parameters are $\bar{a} = -3.67$, $\bar{b} = 2.72$, $\bar{c} = 0.102$ and p value is 0.0909. Figure 6 shows the results. For all of the three numerical experiments, the number of iterations required for convergence for EM algorithms are around 40–50.

To understand the above numerical experiments, we take default events of transportation as an instance. The defaults could help to reveal the economic states, that is X. When there are more defaults occurred, this could give us a hint that economic state is bad, which means $X = 1$. According to $E(X|\mathcal{F}_T)$ under three default intensities in Figs. 4, 5 and 6, all the results are generally consistent with the real-world economic states, i.e., economic situation in US between 1981 and 2002, which demonstrates that our model is able to capture the hidden features and simulate credit default risks. In addition, the interactive Model I with decay effects can better explain the economic situations than those models without decay or decay impacts from other defaults. Actually, from 1981 to 2002, there are several recession periods in the U.S. (2018) : July 1981–Nov 1982, July 1990–Mar 1991 and Mar 2001–Nov 2001. According to the history of economic recession in US, the results of "bad" economic states (recession) shown from Decay Model I are more consistent with the history compared to two other models, which also shows the efficiency and practicability of this model. But on the other hand, we could find that actually Decay Model II with impacts from default events contains more economic information if we consider it with more details. By researching the economy situations in 1990s in US, there is a short-lived economic growth pause during late 1994 and late 1995 because of the increased interest rates raised by Federal Reserve and the affect from the economic financial crises in Mexico in 1995 (1990). This situation is consistent with the results from the Decay Model II in Fig. 5 which indicates bad economic state during this short period.

6 Conclusions

In this paper, we present three different reduced-form intensity-based credit risk models with a hidden Markov process modeling the evolution of economic condition over time. The parameters in these three models are estimated from EM Algorithms with real market data. To illustrate the efficiency of our models with estimated parameters, we conduct empirical experiments and show our results can capture the economic states. Besides, we also take an important credit derivative, CDS, as an example to demonstrate the sensitivity analysis.

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